BI-LOCALLY CONVEX SPACES
AND
SCHAUDER DECOMPOSITIONS

by

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To

My Mother and Father

with

profound respect

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CERTIFICATE

This is to certify that the research work embodied in the present dissertation entitled "Bi-Locally Convex Spaces and Schauder Decompositions" by Mr. Nanda Ram Das, a Ph.D. scholar of this Department, has been carried out under my supervision and that it has not been submitted elsewhere for any degree or diploma.

M Gupta
10/8/82

August - 1982 (*Manjul Gupta*)

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CONTENTS

SYNOPSIS

The theory of two-norm spaces which has its origin in the work of Fichtenholz and of which the structural development goes to Alexiewicz and Semadeni, has been found extremely useful in the study of summability theory, Schauder decompositions etc., for instance one may see the contributions of Orlicz, Subramanian and Rothman. A detailed account of this theory as well as of a closely linked theory, namely, ''Saks Spaces'' is to be seen in a monograph of Cooper: ''Saks Spaces and Applications to Functional Analysis; North Holland Mathematics Studies 28, 1978'', whereas one may also have a glimpse of the same in a recent monograph of Kamthan and Gupta on ''Sequence Spaces and Series; Lecture Notes 65, Marcel Dekker, 1981''.

Although a number of papers have appeared during the last twenty years in the direction of generalizing this theory and the construction of mixed topology due to Wiweger, yet one finds a major gap in all these studies, namely the one created by the absence of the notion of γ-convergence for nets. In the present thesis, we have endeavoured to fill up the same by way of introducing such a notion for nets, and then develop the theory based on this concept for a vector space equipped with two locally convex topologies one being finer than the other, which we call throughout a bi-locally convex space (abbreviated bi-l.c. TVS). We have also investigated its applications in the study of Schauder decompositions (S.D.) of locally convex space vis-a-vis the structural properties of a bi-l.c. TVS arising out of the S.D. and also to vector-valued sequence spaces.

This dissertation entitled ''Bi-locally Convex Spaces and Schauder Decompositions'', contains six chapters.

Chapter one is a collection of some definitions and known results without proof from different texts, monographs and dissertations on the theory of locally convex spaces, Schauder decompositions as well as scalar and vector-valued sequence spaces, which are to be used in the subsequent work of this thesis.

Chapter two sketches a brief history of the development of Saks spaces, two-norm spaces, mixed topology, sequence spaces and Schauder decompositions.

Chapter three is devoted to the structural properties of a bi-l.c. TVS. Indeed, different notions in a bi-l.c. TVS like γ-convergence of nets, normal and quasi-normal bi-l.c. TVS, γ-compactness and γ-boundedness of sets etc. are introduced in the second section of this chapter and a detailed study of these notions is made in the subsequent sections. The last section deals with the relationship among various duals of a bi-l.c. TVS.

Chapter four deals with the duality aspect of the study of a bi-l.c. TVS. In this chapter the notions of γ-conjugate spaces, γ-semireflexivity and γ-reflexivity are introduced and a characterization of γ-semi-reflexivity is also established.

Chapter five incorporates the study of a bi-l.c. TVS arising out of an l.c. TVS with an S.D. which is termed here as a canonical bi-l.c. TVS. The structural properties of such a bi-l.c. TVS are then related with the types of the S.D. Applications of this

study are also investigated in the study of vector-valued sequence spaces.

Chapter six includes the results on the k-reflexivity and the canonical γ-completion of a canonical bi-l.c. TVS. A characterization of a k-reflexive space is established and the relationship of γ-reflexivity with k-reflexivity is further discussed.

CHAPTER - 1

ESSENTIALS

CONTENTS

1. Introduction and Notation :

This chapter serves the purpose of a ready reference for understanding the material embodied in the subsequent chapters of this dissertation. We list here some results from locally convex spaces, Schauder decompositions, and scalar and vector-valued sequence spaces which are quoted without proof as they are available in standard texts, monographs, research papers and theses; for instance one may refer to [57,60,67,69,104,138,140,144]. However, results taken from theses or research papers are preceded or followed by the precise references.

Throughout the sequel we use the following notation :

$\mathbb{R}$ = set of all real numbers

$\mathbb{C}$ = set of all complex numbers

$\mathbb{N}$ = set of all positive integers

$\mathbb{R}_+$ = set of all positive real numbers

$\mathbb{R}^+ = \mathbb{R}_+ \cup \{0\}$

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ equipped with its usual topology

X = A non-trivial vector space over $\mathbb{K}$.

2. Locally Convex Spaces :

Let us begin this section with

Definition 2.1 : A topological vector space (abbreviated TVS) is a pair (X,T) of a linear space X and a topology T such that

the vector addition from $X \times X$ into X and scalar multiplication from $\mathbb{K} \times X$ to X are jointly continuous where the product spaces are equipped with the product topologies defined by T and the usual topology of $\mathbb{K}$. The topology T in this case is referred to a <u>linear</u> or a <u>vector topology</u> of X. It is known that in a TVS (X,T) there exists a fundamental system of neighbourhoods at origin denoted by $\mathcal{U}_T$ consisting of sets which are absorbing, balanced and satisfying the property that for each u in $\mathcal{U}_T$ there exists v in $\mathcal{U}_T$ such that $v+v \subset u$. In addition, if each member of this system is also convex, then the TVS (X,T) is said to be a <u>locally convex topological vector space</u> (abbreviated l.c. TVS) and the corresponding topology T is called a <u>locally convex topology</u>. Throughout, we consider a Hausdorff l.c. TVS.

For $u \in \mathcal{U}_T$, the corresponding Minkowskii functional q_u defined by

$$q_u(x) = \inf \{\alpha : \alpha > 0, \ x \in \alpha u\},$$

is a pseudonorm in case of a linear topology T and is a seminorm for the locally convex topology T. Indeed, in the direction of generation of the topology T, by family of such functions, we have

<u>Proposition</u> 2.2 : A vector space X equipped with a topology T is a TVS (resp. l.c. TVS) if and only if there exists a family of pseudonorms (resp. seminorms) on X generating a unique topology equivalent to T.

<u>NOTE</u> : We shall reserve the symbol $\mathcal{D}_T$ or $\mathcal{D}$ to denote the famil[y] of all continuous pseudonorms or seminorms generating the

topology T. If there are two topologies T_1 and T_2, we write $\mathcal{D}_{T_1}$ and $\mathcal{D}_{T_2}$ for the corresponding family of pseudonorms or seminorms.

If T_1 is finer than T_2 (or T_2 is coarser than T_1), we write $T_2 \subset T_1$ which is equivalent to the fact that for each $q \in \mathcal{D}_{T_2}$ there exists a $p \in \mathcal{D}_{T_1}$ and $M \in \mathbb{R}_+$ such that

$$q(x) \leq M\, p(x), \ \forall\, x \in X.$$

If T_1 and T_2 are topologies such that $T_2 \subset T_1$ and $T_1 \subset T_2$, then T_1 and T_2 are said to be equivalent topologies and we denote it by $T_1 \approx T_2$.

If (X,T) is a topological space and $Y \subset X$, then the topology induced on Y by T is denoted by $T|_Y$.

The continuity and equicontinuity of linear maps are characterized in terms of pseudonorms as follows.

Proposition 2.3 : Let (X,T) and (Y,T_1) be two TVS.

(i) A linear map $R : (X,T) \to (Y,T_1)$ is continuous if to each $q \in \mathcal{D}_{T_1}$ there exists $p \in \mathcal{D}_T$ and $M \in \mathbb{R}_+$ such that

$$q(R(x)) \leq M\, p(x), \quad \forall\, x \in X.$$

(ii) A family H of linear maps $R : (X,T) \to (Y,T_1)$ is equicontinuous family if to each $q \in \mathcal{D}_{T_1}$, there exists $p \in \mathcal{D}_T$ and $M \in \mathbb{R}_+$ such that

$$q(R(x)) \leq M\, p(x) \quad , \quad \forall\, x \in X,\ R \in H.$$

We also need the following simple facts in a TVS.

Proposition 2.4 : (i) A subset B in a TVS (X,T) is bounded if and only if $\varepsilon_n x_n \to 0$ in (X,T) whenever $\{x_n\}$ is a sequence in B and $\varepsilon_n \to 0$ in $\mathbb{K}$.

(ii) A continuous linear mapping f from a TVS (X,T) into another TVS (Y,T_1) maps a bounded subset into a bounded subset.

We now state the well known Hahn-Banach theorem along with its important consequences in the following results

Theorem 2.5 : Let M be a linear subspace of an l.c. TVS (X,T) and f be a continuous linear functional on M. Then there exists a continuous linear functional $\hat{f}$ on X such that $f(x) = \hat{f}(x)$ for all $x \in M$.

Theorem 2.6 (Mazur) : Let (X,T) be an l.c. TVS and M a closed, convex and balanced subset of X. Then for $x_o \notin M$, there exists a continuous linear functional f on X such that $f(x_o) > 1$ and $|f(x)| \leq 1$ on M.

Proposition 2.7: Let X be an l.c. TVS, M a closed subspace of X and $x_o \in X$ such that $x_o \notin M$. Then there exists a continuous linear functional f on X such that $f(x_o) = 1$ and $f(x) = 0$ for all $x \in M$.

Proposition 2.8 : Let X be an l.c. TVS, x_o a point in X and p a continuous seminorm on X. Then there exists a continuous linear functional f on X such that $f(x_o) = p(x_o)$ and $|f(x)| \leq p(x)$ on X.

Coming to the duals of an l.c. TVS (X,T), let us mention that the symbols X', X^+ and X^* stand respectively for the algebraic, sequential and topological dual of X. Clearly,

$$X^* \subset X^+ \subset X'.$$

Following ([60], p. 29), we have

<u>Definition</u> 2.9 : An l.c. TVS (X,T) is said to be a <u>Mazur space</u> if $X^* = X^+$.

<u>Duality</u> :

In order to have an insight of the duality theory displayed by an l.c. TVS and its duals, we consider in this subsection a dual pair $\langle X,Y\rangle$ of vector spaces X and Y over the same field $\mathbb{K}$ and some results relevant to our study in this dissertation. A <u>dual pair</u> $\langle X,Y\rangle$ of vector spaces X and Y is a pairing for which there is defined a bi-linear form $B : X \times Y \to \mathbb{K}$, denoted by $B(x,y) = \langle x,y\rangle$, $x \in X$, $y \in Y$ which separates points of both the spaces X and Y. The natural locally convex topology on X (resp. on Y) generated by the family $\{q_y : y \in Y\}$ (resp. $\{q_x : x \in X\}$), where $q_y(x) = |\langle x,y\rangle|$ (resp. $q_x(y) = |\langle x,y\rangle|$) is called the <u>weak topology</u> on X (resp. on Y) and is denoted by $\sigma(X,Y)$ [resp. $\sigma(Y,X)$]. For a subset A of X, the polar of A, denoted by A°, is a subset of Y defined by

$$A^\circ = \{y \in Y : |\langle x,y\rangle| \leq 1, \quad \forall\, x \in A\}$$

If A is a $\sigma(Y,X)$-bounded subset of Y, then A° is absorbing, balanced and convex and therefore corresponding to a collection

$\mathcal{S}$ of $\sigma(Y,X)$ bounded sets in Y, we can generate a locally convex topology on X with the help of the system $\{A^\circ : A \in \mathcal{S}\}$, or equivalently by the family of seminorms $\{p_A : A \in \mathcal{S}\}$, where

$$p_A(x) = \sup_{y \in A} |\langle x,y\rangle|, \quad x \in X.$$

The topology thus obtained is called an $\mathcal{S}$-<u>topology</u> or a <u>polar topology</u> on X. In case $\mathcal{S}$ is the collection of all finite subsets of Y, all $\sigma(Y,X)$-bounded subsets of Y and all balanced, convex and $\sigma(Y,X)$-compact subsets of Y, then the respective $\mathcal{S}$-topologies on X are the <u>weak</u>, <u>strong</u> and <u>Mackey</u> topologies denoted respectively by $\sigma(X,Y)$, $\beta(X,Y)$ and $\tau(X,Y)$. Similarly, we can define $\mathcal{S}$-topologies $\sigma(Y,X)$, $\beta(Y,X)$ and $\tau(Y,X)$ on Y.

Next we have

<u>Definition</u> 2.10 : A locally convex topology T on X is said be <u>compatible</u> with the dual pair $\langle X,Y\rangle$ if the topological dual of X relative to T is Y.

<u>Proposition</u> 2.11 : Let $\langle X,Y\rangle$ be a dual pair. The bounded (resp. closed convex) subsets of X are the same for all locally convex topologies on X compatible with the dual pair $\langle X,Y\rangle$.

<u>Theorem</u> 2.12 (Bi-polar) : For a dual pair $\langle X,Y\rangle$ if A is a non-empty subset of X, then $A^{\circ\circ}$ is the smallest balanced, convex, $\sigma(X,Y)$-closed set containing A.

Following is from [60] (cf.also [140], p. 158)

<u>Definition</u> 2.13 : A dual pair $\langle X,Y\rangle$ is said to be an M-<u>system</u> (or, <u>Banach-Mackey</u>) if every $\sigma(Y,X)$-bounded set is $\beta(Y,X)$-bounded

Concerning an M-system, we have

<u>Proposition</u> 2.14 : If a dual pair <X,Y> is an M-system, so is also <Y,X>. Further, for every sequentially complete l.c. TVS(X,T) <X,X^*> is an M-dual system.

Concerning subsets of the dual, we have

<u>Proposition</u> 2.15 : Let X be a TVS. (i) A subset M of X^* is equicontinuous if and only if $M \subset v^\circ$ for some $v \in \mathcal{U}_T$. (ii) Any equicontinuous subset of X^* is relatively compact for the topology $\sigma(X^*,X)$.

The statement (ii) of this proposition is known as <u>Alaoglu Bourbaki theorem</u> in the literature.

Different types of l.c. TVS required in the sequel are defined in

<u>Definition</u> 2.16 : An l.c. TVS is said to be (i) <u>barrelled</u> space if every barrel (i.e., an absorbing, balanced, convex and closed subset) in X is a neighbourhood of origin; (ii) <u>infrabarrelled</u> if every bornivorous (i.e., a subset which absorbs every bounded subset of X) barrel in X is a neighbourhood of origin; (iii) σ-<u>barrelled</u> (resp. σ-<u>infrabarrelled</u>) if every countable $\sigma(X^*,X)$-bounded (resp. $\beta(X^*,X)$-bounded) subset of X^* is equicontinuous; (iv) <u>bornological</u> if every balanced, convex, bornivorous subset of X is a neighbourhood of origin.

<u>Note</u> : Every bornological space is a Mazur space since $X^* = X^+$ for such spaces.

Definition 2.17 : A TVS X is said to be an S-space if $(X^*, \sigma(X^*,X))$ is sequentially complete.

Definition 2.18 : A subset A of X^* for an l.c. TVS X is said to be nearly closed if $A \cap u^o$ is $\sigma(X^*,X)$-closed for every neighbourhood u in X. An l.c. TVS is said to be fully complete if every nearly closed vector subspace of X^* is $\sigma(X^*,X)$-closed.

We also need the following characterizations of barrelled and infrabarrelled spaces.

Proposition 2.19 : An l.c.TVS X is barrelled if and only if every $\sigma(X^*,X)$-bounded subset of X^* is equicontinuous.

Proposition 2.20 : An l.c.TVS X is infrabarrelled if and only if every $\beta(X^*,X)$-bounded subset of X^* is equicontinuous. A sequentially complete infrabarrelled space is a barrelled space.

We quote from [55]

Proposition 2.21 : If (X,T) is a Mazur space then its strong dual $(X^*,\beta(X^*,X))$ is complete.

Next, we have

Proposition 2.22 : Let (X,T) be a Mazur space and T_1 be a locally convex topology compatible with and coarser than T. Then (X,T) is Mazur.

Concerning linear maps, we have

Proposition 2.23 : A linear map f from a bornological space X to an arbitrary l.c. TVS Y is continuous provided it maps bounded sets into bounded sets.

<u>Theorem</u> 2.24 : (Banach-Steinhaus) Let X be a barrelled space, Y an l.c. TVS, $\{f_n\}$ a sequence of continuous linear maps and $g : X \to Y$ a linear map such that $f_n \to g$ pointwise on X. Then g is continuous.

<u>Theorem</u> 2.25 : (Open mapping theorem) A continuous linear mapping from a fully complete space onto a Hausdorff barrelled space is open. Consequently, a continuous, one-to-one, linear mapping of a fully complete space onto a Hausdorff barrelled space is an isomorphism.

<u>Reflexivity</u> :

For an l.c. TVS (X,T), the second dual X^{**} is the topological dual of $(X^*, \beta(X^*,X))$; and the map $J : X \to X^{**}$ defined by

$$(Jx)(f) = f(x), \quad \forall\, f \in X^*$$

is an one-to-one, linear map from X to X^{**}, which is usually referred to the canonical <u>embedding</u> of X into X^{**}. If the map J is onto, the l.c. TVS (X,T) is said to be <u>semi-reflexive</u> and in case J is a topological isomorphism from (X,T) onto $(X^{**},\beta(X^{**},X^*))$, (X,T) is said to be <u>reflexive</u>.

Concerning these spaces, we have

<u>Proposition</u> 2.26 : An l.c. TVS (X,T) is semireflexive if and only if every bounded, $\sigma(X,X^*)$-closed subset of X is $\sigma(X,X^*)$-compact.

Proposition 2.27 : The topological dual X^* of a semireflexive locally convex space X is barrelled relative to the topology $\beta(X^*,X)$.

Proposition 2.28 : An l.c. TVS is reflexive if and only if it is semireflexive and infrabarrelled.

Proposition 2.29 : For a semireflexive space (X,T), the dual pair $\langle X,X^* \rangle$ is an M-system.

Adjoint maps :

Let (X,T) and (Y,S) be two l.c. TVS and R be a linear map from X to Y. Then we know that its transpose is a linear map from Y' to X'. If R is also continuous, then the transpose map transforms Y^* into X^* and is called the adjoint of R, denoted by R^* and defined by

$$[R^*(f)](x) = f(R(x)), \ x \in X, \ f \in Y^*.$$

Following are some useful facts concerning these adjoint maps.

Proposition 2.30 : If a linear map R from an l.c. TVS (X,T) into another l.c. TVS (Y,S) is continuous then (i) R is $\sigma(X,X^*) - \sigma(Y,Y^*)$ continuous and (ii) its adjoint $R^* : Y^* \to X^*$ is $\sigma(Y^*,Y) - \sigma(X^*,X)$ continuous and also $\beta(Y^*,Y) - \beta(X^*,X)$ continuous.

Proposition 2.31 : Let X be an l.c. TVS and M a subspace of X. Then the topology $\sigma(M,M^*)$ coincides with the topology induced on M by $\sigma(X,X^*)$.

3. Schauder Bases and Decompositions :

This section provides necessary background from [61], [74],[102] and [112]. We begin with

Definition 3.1 : A sequence $\{x_i\}$ in a TVS (X,T) is said to be a basis for X if for each x in X, there is a unique sequence $\{\alpha_i\}$ of scalars such that

$$x = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i x_i$$

where the limit is taken relative to the topology T. The unique representation of each x in X in terms of $\{x_i\}$ yields a sequence $\{f_i\}$ of linear functionals on X known as the sequence of associated co-ordinate functionals (s.a.c.f.) defined as follows

$$f_i(x) = \alpha_i, \ i \geq 1$$

A basis $\{x_i\}$ is said be a Schauder basis if each f_i is continuous. Whenever we want to emphasize the s.a.c.f., we write $\{x_i;f_i\}$ for a Schauder basis.

For our subsequent work, we need the following types of Schauder bases

Definition 3.2 : A Schauder basis $\{x_i;f_i\}$ for (X,T) is called (i) e-Schauder basis if $S_n = \sum_{i=1}^{n} f_i(x)\ x_i$, $n \geq 1$ is T-T equi-continuous; (ii) monotone if $p(\sum_{i=1}^{m} f_i(x)x_i) \leq p(\sum_{i=1}^{n} f_i(x)x_i)$ for each $p \in \mathcal{D}_T$ and all $m,n \in \mathbb{N}$ with $m \leq n$; (iii) unconditional if for each $x \in X$, the series $\sum_{i\geq 1} f_i(x)\ x_i$ converges unconditionall

to x; (iv) <u>shrinking</u> if $\{f_i\}$ is a Schauder basis for $(X^*,\beta(X^*,X))$ and (v) <u>boundedly complete</u> if $\sum_{i\geq 1} \alpha_i x_i$ converges to a point of X, whenever, $\{\sum_{i=1}^{n} \alpha_i x_i : n \geq 1\}$ is bounded for a sequence $\{\alpha_i\}$ of scalars.

The notion of a Schauder base in a TVS is generalized to yield

<u>Definition</u> 3.3 : A sequence $\{M_i\}$ of non-trivial subspaces of a TVS (X,T) is called a <u>decomposition</u> provided for each $x \in X$ there exists a unique sequence $\{x_i\}$ in X with $x_i \in M_i$, $i \geq 1$ such that

$$x = \lim_{n\to\infty} \sum_{i=1}^{n} x_i$$

the limit being considered in the topology T of X. Corresponding to a decomposition $\{M_i\}$ of X, we can define a sequence $\{P_i\}$ of projection maps P_i from X to M_i, such that $P_i(x) = x_i$, $i \geq 1$. We say $\{M_i\}$ is a <u>Schauder decomposition</u> of (X,T) if each P_i is T-T continuous and we write $\{M_i;P_i\}$ for a Schauder decomposition (S.D.) $\{M_i\}$.

We write

$$S_n = \sum_{i=1}^{n} P_i, \ n \geq 1.$$

Also, we use the symbols P_n^*, S_n^* for adjoint maps of P_n and S_n respectively, I_{X^*} for the identity operator on X^* and $V_n^* = I_{X^*} - \sum_{i=1}^{n} P_i^*$, $n \geq 1$.

Following are several types of Schauder decomposition, useful in the sequel.

Definition 3.4 : An S.D. $\{M_n;P_n\}$ of a TVS (X,T) is said to be (i) an e-Schauder decomposition (e-S.D) if the sequence $\{S_n\}$ is T-T equicontinuous; (ii) monotone if for each $p \in \mathcal{D}_T$ and $x = \sum_{i\geq 1} x_i$, $x_i \in M_i$, $i \geq 1$

$$p(\sum_{i=1}^{m} x_i) \leq p(\sum_{i=1}^{n} x_i)$$

for all m,n with $m \leq n$; (iii) boundedly complete if for each sequence $\{x_i\}$ with $x_i \in M_i$, $i \geq 1$, and $\{\sum_{i=1}^{n} x_i : n \geq 1\}$ bounded in X, the series $\sum_{i\geq 1} x_i$ is convergent in X; (iv) shrinking at $f \in X^*$ if

$$\lim_{n\to\infty} V_n^*(f) = 0 \text{ in } \beta(X^*,X),$$

or, equivalently, $V_n^*(f)$ converges to zero uniformly on each T-bounded subset of X; (v) shrinking if it is shrinking at each $f \in X^*$; and (vi) τ-uniform for a linear topology τ on X if for each T-bounded set B of X, $\sum_{i=1}^{n} P_i(x)$ converges to x uniformly in $x \in B$ for the topology τ.

Concerning Schauder basis, we have from [55]

Proposition 3.5 : A barrelled space having a Schauder basis is a Mazur space.

Let us now recall the following results respectively from [49], [111], [58] and [59].

Proposition 3.6 : An S.D. $\{M_n;P_n\}$ for an l.c. TVS (X,T) is shrinking if and only if $\{R(P_i^*);P_i^*\}$ is a $\beta(X^*,X)$-S.D of X^*.

Proposition 3.7 : If $\{M_n;P_n\}$ is an e-S.D. of an l.c. TVS (X,T) then the topology T is equivalent to a locally convex topology $\bar{T}$ on X which is generated by the family $\{\bar{p}_\alpha\}$ of seminorms, where $\bar{p}_\alpha(x) = \sup_{n\geq 1} p_\alpha(S_n(x))$ for $p_\alpha \in \mathcal{D}_T$; and the S.D. $\{M_n;P_n\}$ is monotone in $(X,\bar{T})$.

Proposition 3.8 : Let $\{M_n\}$ be an e-S.D. for an l.c. TVS (X,T). Then X is complete (resp. quasicomplete, resp. sequentially complete) if and only if

(i) each M_n is complete (resp. quasicomplete, resp. sequentially complete) and

(ii) $\{M_n\}$ is a complete decomposition.

Proposition 3.9 : Let X be an l.c. TVS with an S.D. $\{M_n;P_n\}$. Then

(i) if X is barrelled, each M_n is barrelled

(ii) if X is semireflexive, each M_n is semi-reflexive.

4. Sequence Spaces :

We bifurcate this section into two parts, namely, scalar valued sequence spaces (SVSS) and vector valued sequence spaces (VVSS). For the theory of SVSS, we refer to monographs [60] and [61]; whereas results on VVSS are to be found in [33], [51] [52] and [91].

Scalarvalued sequence spaces :

Let us begin this subsection with the symbols ω and ϕ which respectively stand for the vector space of all scalar valued sequences and the vector space of all finitely non-zero sequences relative to the pointwise addition and scalar multiplication. A sequence space λ is a subspace of ω, containing ϕ.

Let $e^i = \{0,0,\ldots,0,1,0,\ldots\}$, 1 being in the ith co-ordinate and $e = \{1,1,\ldots,1,\ldots\}$; e^i and e are respectively called the <u>i-th</u> <u>unit vector</u> and the <u>unity</u> of ω. Clearly $\phi = \text{sp}\,\{e^n : n \geq 1\}$. Some commonly used sequence spaces along with their natural norms, which we use in the sequel are given below

Sequence Spaces	Normed by
$c_o = \{\{\alpha_i\} \in \omega : \lim\limits_n \alpha_n = 0\}$	$\|\{\alpha_i\}\|_\infty = \sup \|\alpha_i\|$
$c = \{\{\alpha_i\} \in \omega : \lim\limits_n \alpha_n$ exists in $\mathbb{K}\}$	$\|\{\alpha_i\}\|_\infty = \sup \|\alpha_i\|$
$\ell^\infty = \{\{\alpha_i\} \in \omega : \sup\limits_n \|\alpha_n\| < \infty\}$	$\|\{\alpha_i\}\|_\infty = \sup \|\alpha_i\|$
$m_o = \text{sp}\{A\}$, where A is the set of all sequences of zeros and ones	$\|\{\alpha_i\}\|_\infty = \sup \|\alpha_i\|$
$\ell^1 = \{\{\alpha_i\} \in \omega : \sum\limits_{i\geq 1} \|\alpha_i\| < \infty\}$	$\|\{\alpha_i\}\|_1 = \sum\limits_{i\geq 1} \|\alpha_i\|$
$bv = \{\{\alpha_i\} \in \omega : \lim\limits_n \sum\limits_{i=1}^{n} \|\alpha_{i+1} - \alpha_i\|$ exists$\}$	$\|\{\alpha_i\}\|_{bv} = \sum\limits_{n=1}^{\infty} \|\alpha_{i+1} - \alpha_i\| + \lim\limits_i \|\alpha_i\|$.

<u>Note</u> : Elements of an SVSS λ are denoted by $\bar{\alpha} = \{\alpha_i\}, \bar{\beta} = \{\beta_i\}$ etc.

For a sequence space λ, we define its <u>Köthe dual</u> or α-<u>dual</u> $\lambda^\times$ as follows :

$$\lambda^\times = \{\{\alpha_i\} \in \omega : \sum_{i\geq 1} |\alpha_i\beta_i| < \infty, \forall \{\beta_i\} \in \lambda\}.$$

One can easily check that the sequence spaces λ and $\lambda^\times$ form a dual system with respect to the bi-linear form

$$B(\bar{\alpha},\bar{\beta}) = \sum_{i\geq 1} \alpha_i\beta_i,$$

$\bar{\alpha} \in \lambda$ and $\bar{\beta} \in \lambda^\times$. Thus one can equip λ and $\lambda^\times$ with natural polar topologies, for instance, the weak topology $\sigma(\lambda,\lambda^\times)$, Mackey topology $\tau(\lambda,\lambda^\times)$ and strong topology $\beta(\lambda,\lambda^\times)$ on λ. Other than these topologies, there are several more locally convex topologies on λ which are defined corresponding to a system S of balanced, convex, $\sigma(\lambda^\times,\lambda)$-compact and normal subsets A (A is normal if $\bar{\beta} \in A$ whenever $|\beta_i| \leq |\alpha_i|$ for some $\bar{\alpha} \in A$) of $\lambda^\times$ covering $\lambda^\times$, generated by the family $\{q_A : A \in S\}$ of seminorms, where

$$q_A(\bar{\alpha}) = \sup_{\bar{\beta}\in A} \sum_{i\geq 1} |\alpha_i\beta_i|$$

These topologies are known as the <u>solid</u> S-topologies on λ. In particular, if S contains the normal hulls of singletones, the corresponding topology is known as the <u>normal</u> topology and denoted by $\eta(\lambda,\lambda^\times)$ and is also generated by $\{q_{\bar{\beta}} : \bar{\beta} \in \lambda^\times\}$ where

$$q_{\bar{\beta}}(\bar{\alpha}) = \sum_{i\geq 1} |\alpha_i\beta_i|, \ \bar{\alpha} \in \lambda .$$

Next, we have

<u>Definitions</u> 4.1 : A sequence space λ is said to be (i) <u>normal</u> if $\bar{\alpha} \in \lambda$ whenever $|\alpha_i| \leq |\beta_i|$, $i \geq 1$, for some $\bar{\beta} \in \lambda$; (ii) <u>perfect</u> if $\lambda = \lambda^{\times\times}$; and (iii) a sequence space λ equipped with a linear topology T is said to be a K-<u>space</u> if the co-ordinate maps P_i defined from λ to $\mathbb{K}$ by $P_i(\bar{\alpha}) = \alpha_i$, $i \geq 1$, are <u>continuous</u>. A K-space (λ,T) is an FK-<u>space</u> if T is a Fréchet topology on λ.

With this background on sequence spaces, we now state the results given in [60]

<u>Proposition</u> 4.2 : $\sigma(\omega,\phi) = \eta(\omega,\phi) = \beta(\omega,\phi)$ and each of the above topologies is the same as the topology of co-ordinatewise convergence on ω.

<u>Proposition</u> 4.3 : (i) Sequential convergence in ℓ^1 relative to $\sigma(\ell^1,m_o)$ and the usual norm topology on ℓ^1 coincide. Consequently, $(\ell^1, \sigma(\ell^1,m_o))$ is sequentially complete.

(ii) $(\ell^1, \tau(\ell^1,c_o))$ is sequentially barrelled and hence a Banach-Mackey space but is not an S-space; also this space is not σ-infrabarrelled and hence not σ-barrelled.

(iii) $\beta(\ell^1,\phi)$ is the norm topology of ℓ^1 given by $||\cdot||_1$.

(iv) A subset B of ℓ^1 is $\sigma(\ell^1,\ell^\infty)$-compact if and only if it is $\beta(\ell^1,\ell^\infty)$-compact. Also, a $\sigma(\ell^1,\ell^\infty)$ or $\beta(\ell^1,\ell^\infty)$ bounde

subset B of ℓ^1 in $\sigma(\ell^1,\ell^\infty)$ or $\beta(\ell^1,\ell^\infty)$-relatively compact if and only if

$$\lim_{n\to\infty} \sup_{x\in B} \sum_{i\geq n} |x_i| = 0.$$

Proposition 4.4 : (i) $(\ell^\infty, \tau(\ell^\infty,\ell^1))$ is a Banach-Mackey space without being infrabarrelled.

(ii) m_o is dense in ℓ^∞ relative to its norm topology.

Proposition 4.5 : Let λ be a sequence space and μ be a normal subspace of λ^x. Then $\sigma(\lambda,\mu)$ and $\eta(\lambda,\mu)$ convergent sequences in λ are the same.

Theorem 4.6 : Let (λ,T) be a sequence space equipped with a linear topology T which is finer than the topology of co-ordinatewise convergence on λ. For a subset M of λ, the following statements are equivalent.

(i) M is T-compact (resp. T-relatively compact).

(ii) M is T-sequentially compact (resp. T-relatively sequentially compact).

(iii) M is T-countably compact (resp. T-relatively countably compact)

(iv) M is compact in the topology of coordinatewise convergence, and T and the topology of coordinatewise convergence give rise to the same convergent sequences in M (resp., M is coordinatewise bounded, and any sequen of points of M convergent coordinatewise in ω converg to a point of λ in the topology T)

Vector-valued sequence spaces :

The vector-valued sequence spaces (VVSS) are natural generalization of SVSS. Indeed, for a dual pair $\langle X,Y\rangle$ of vector spaces a vector-valued sequence space is a vector space $\Lambda(X)$ of sequences from X with respect to the usual pointwise addition and scalar multiplication. The generalized Köthe dual $\Lambda^{\times}(Y)$ of $\Lambda(X)$ is the space defined by

$$\Lambda^{\times}(Y) = \{\{y_i\} : y_i \in Y,\ i \geq 1 \text{ and } \sum_{i\geq 1} |\langle x_i, y_i\rangle| < \infty \text{ for all } \{x_i\} \in \Lambda(X)\}$$

If $\Phi(X)$ denotes the vector space of all sequences $\{x_i\} \subset X$ with $x_i = 0$ for all but a finite number of indices i, and $\Phi(X) \subset \Lambda(X)$ then $\langle \Lambda(X), \Lambda^{\times}(Y)\rangle$ forms a dual pair with respect to the bilinear form B, $B(\bar{x},\bar{y}) = \sum_{i\geq 1} \langle x_i, y_i\rangle$. Thus, analogous to scalar case, we can define weak, strong and Mackey topologies on either of the spaces $\Lambda(X)$ and $\Lambda^{\times}(Y)$. Also, there is a natural locally convex topology on $\Lambda(X)$, known as the normal topology which is denoted by $\eta(\Lambda(X), \Lambda^{\times}(Y))$ and is generated by the family $\{p_{\bar{y}} : \bar{y} \in \Lambda^{\times}(Y)\}$ of seminorms defined by

$$p_{\bar{y}}(\bar{x}) = \sum_{i\geq 1} |\langle x_i, y_i\rangle|$$

for each $\bar{y} = \{y_i\} \in \Lambda^{\times}(Y)$ and $\bar{x} = \{x_i\} \in \Lambda(X)$. We reserve $\bar{x}, \bar{y}$ etc. to denote a vector-valued sequence, that is, $\bar{x} = \{x_i\}$. For $x \in X$, the symbol δ_i^x stands for the sequence $\{0,0,\ldots,0,x,0,\ldots$ where x is placed at the i-th co-ordinate. The n-th section $\bar{x}^{(r}$ of $\bar{x}$ is defined to be the sequence $\{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}$. Clea

$$\bar{x}^{(n)} = \sum_{i=1}^{n} \delta_i^{x_i}.$$

In the sequel we consider throughout VVSS $\Lambda(X)$ containing $\Phi(X)$.

Let us now take X to be an l.c. TVS and equip $\Lambda(X)$ with a Hausdorff locally convex topology $\mathcal{G}$. Then we have from [91] (cf. also [51] and [52])

Definition 4.7 : A VVSS $(\Lambda(X), \mathcal{G})$ is called a GK-space (resp. GC-space) if the maps $C_i : \Lambda(X) \to X$, $C_i(\bar{x}) = x_i$, $i \geq 1$ and $\bar{x} = \{x_i\} \in \Lambda(X)$ (resp. the maps $R_i : X \to \Lambda(X)$, $R_i(x) = \delta_i^x$, $x \in X$, $i \geq 1$, are continuous; and a GK-space $(\Lambda(X), \mathcal{G})$ is said to be a GAK-space if for each $\bar{x} \in \Lambda(X)$, $\bar{x}^{(n)} \to \bar{x}$ in $\mathcal{G}$.

For $i \geq 1$, we define

$$N_i = \{\delta_i^x : x \in X\}.$$

Clearly, each N_i is a subspace of $\Lambda(X)$ and the natural projection maps P_i, $i \geq 1$ defined from $\Lambda(X)$ to N_i are given by

$$P_i(\bar{x}) = \delta_i^{x_i}, \quad \bar{x} = \{x_i\} \in \Lambda(X), \ i \geq 1.$$

Now, we quote from [54]

Theorem 4.8 : $\{N_i; P_i\}$ is an S.D. for a VVSS $\Lambda(X)$ relative to the topology $\sigma(\Lambda(X), \Lambda^{\times}(X^*))$, where X is an l.c. TVS with dual X^*. Further, if $\{N_i\}$ is an e-S.D. for $(\Lambda(X), \sigma(\Lambda(X), \Lambda^{\times}(X^*))$, then $\Lambda^{\times}(X^*) = \Phi(X^*)$.

Another type of a VVSS which we consider in the sequel is defined corresponding to a perfect SVSS λ (cf. [33]). In fact corresponding to an SVSS λ and an l.c. TVS (X,T), we set

$$\lambda(X) = \{\{x_i\} : \{p(x_i)\} \in \lambda, \quad \text{for each } p \in \mathcal{D}_T\}.$$

If λ is equipped with a solid $\mathcal{S}$-topology defined in the preceding subsection, we topologize $\lambda(X)$ with a locally convex topology $\mathfrak{I}_\lambda$, generated by the family $\{Q_{A,p} : A \in \mathcal{S}$ and $p \in \mathcal{D}_T\}$, of seminorms where

$$Q_{A,p}(\bar{x}) = q_A(\{p(x_i)\}) = \sup_{\bar{\beta} \in A} \sum_{i \geq 1} p(x_i) \, |\beta_i|$$

Concerning this space $(\lambda(X), \mathfrak{I}_\lambda)$, De Grande-De Kimpe [33] has proved

Proposition 4.9 : $(\lambda(X), \mathfrak{I}_\lambda)$ is a GK-, GAK- and GC-space.

CHAPTER - 2

HISTORICAL GLIMPSES

CONTENTS

1. Two-Norm Spaces:

The theory of two-norm spaces, an invention of the Polish School of Mathematics was essentially founded by Fichtenholz [40] around the year 1938, who introduced in some concrete Banach spaces a kind of convergence weaker than that generated by the given norm. Indeed his results led Alexiewicz [1] to introduce the notion of γ-convergence or two-norm convergence of sequences in a vector space X equipped with two norms $||\cdot||$ and $||\cdot||^*$ and develop the theory of such spaces in collaboration with Semadeni in a series of papers [14, 15, 16] which was later on found useful in the theory of sequence spaces [60] and the theory of Schauder decompositions (S.D.) in Banach spaces [135, 136, 137]. Almost simultaneously, Orlicz [85] motivated by the work of Saks [113, 114] introduced the notion of ℓ-convergence (which is the same as γ-convergence and means the $||\cdot||^*$-convergence of a $||\cdot||$-bounded sequence in X) and studied a closely linked theory, namely that of Saks spaces which he applied to the theory of linear methods of summability [86, 88, 89]. In the following few pages, we shall give a brief account of the chronological development of these two parallel theories and shall also touch upon their applications at a later stage.

Around the year 1948, Orlicz [85] considered the unit ball B of a normed space $(X, ||\cdot||)$ equipped with another norm $||\cdot||^*$ such that $(B, ||\cdot||^*)$ is complete and he termed the same as Saks space which we shall denote by X_S in the sequel. In this

paper he presented several examples of Saks spaces satisfying certain convergence conditions and obtained sufficient conditions for the continuity and boundedness of linear operators defined on Saks spaces. In his subsequent paper [86], Orlicz tackled the problem of convergence and divergence of the sequence $\{T_n\}$ of linear operators defined on a Saks space X_S and derived an analogue of the well-known Banach-Steinhaus theorem for operators defined from Saks spaces to Banach spaces. He also dealt with some problems related to the theory of summability and Saks spaces.

Later in 1957, Orlicz in collaboration with Matuszewska [75] and Pták [90] solved some problems involving the dual of a Saks space (that is, dual of $(B, ||\cdot||^*)$). Indeed, in [75] the authors dealt with a concrete Saks space $(M_S^{g\phi}, ||\cdot||, ||\cdot||^*)$ of essentially bounded and measurable real-valued functions equipped with two suitable norms and established a general representation theorem for continuous linear functionals on the Saks space $M_S^{g\phi}$. However, in [90] Orlicz and Pták proved that the dual of $(B, ||\cdot||^*)$ is a closed subspace of the dual of the space $(X, ||\cdot||)$ They also characterized the continuity of a linear mapping from a Saks space to a separable Banach space in terms of the members of the dual of $(B, ||\cdot||^*)$. In the same year Orlicz [88] generalized some of his earlier results appeared in [86], in order to apply the same to the theory of summability, whereas in [87] he gave necessary and sufficient conditions for the equivalence

of the two norms $||\cdot||$ and $||\cdot||^*$, related the $||\cdot||^*$-completeness of the Saks space with the equality of the dual of $(B,||\cdot||^*)$ and $(X,||\cdot||^*)$ and dealt with the problem of representing the members of $X_S^* = (B, ||\cdot||^*)^*$ in terms of members of X_1^* and X_2^* where $X_1^* = (X,||\cdot||)^*$ and $X_2^* = (X,||\cdot||^*)^*$. Apart from giving some meaningful examples of Saks spaces in incomplete normed spaces, he utilized the general theory of Saks spaces to examine the continuity and boundedness of linear operators from Saks spaces to Banach spaces. A brief sketch of these developments since the appearance of Sak's paper [113] in 1932 together with some unsolved problems is to be found in Orlicz's paper [89] published in 1959.

After a lapse of almost two decades, there appeared a generalization of this theory, replacing the range space of linear operators acting on a Saks space, from Banach or Fréchet spaces to an arbitrary TVS or an l.c.TVS. Indeed in 1974, I. Labuda [71] discovered that the proofs of the original results of Orlicz [85] are more direct in the case of l.c.TVS or TVS; and in view of this, Labuda refined some classical results of Orlicz [85]. Motivated by the developments in the theory of vector measures, he also obtained several new results from a vector measure to an additive operator on a Saks space. In the same year, Labuda in collaboration with Orlicz [73] introduced the notion of Saks set which according to them, means the unit ball X_S of a Banach space $(X,||\cdot||)$ equipped with the

topology induced on X_S generated by a complete metric d on X; and a Saks space if X_S is complete with respect to d. They proved if d gives rise to a complete locally convex topology on separable Banach space X, then d generates a norm topology on X provided the corresponding Saks set satisfies certain conditions in terms of d-continuous linear functionals. Continuing his study of Saks spaces, Labuda [72] in 1975 characterized the existence of non-trivial Saks spaces (sets) satisfying certain conditions in a Banach space X.

Coming to the closely associated theory of two-norm spaces, let us once again go back to the year 1950 when Alexiewicz introduced the notion of two-norm convergence or γ-convergence in [1]. According to him, a sequence $\{x_n\}$ in a vector space X equipped with a norm $||\cdot||$ and an F-norm $||\cdot||^*$ is said to be γ-convergent to a point x in X if $\{x_n\}$ is $||\cdot||$-bounded and $||x_n - x||^* \to 0$ as $n \to \infty$. The triplet $<X, ||\cdot||, ||\cdot||^*>$ is called a two-norm space and is denoted once again by X_S. The author proved that the γ-convergence satisfies several interesting properties which are useful for solving a set of problems posed by Mazur and Orlicz [76] in 1933 concerning the study of linear operators from one vector space to another equipped with linear sequential convergence and in particular γ-γ continuous linear operators [A linear operator R from one two-norm space to another is said to be γ-γ continuous if $x_n \xrightarrow{\gamma} x \Longrightarrow R(x_n) \xrightarrow{\gamma} R(x)$].

In his subsequent work [2], Alexiewicz continued the study of two-norm convergence and mainly dealt with the problem of γ-continuity of the pointwise limit of a sequence of γ-continuous linear functionals. Besides characterizing γ-convergence as a norm convergence in terms of several equivalent conditions, he mentioned several examples of two-norm spaces which are sequentially complete relative to γ-convergence and normal (i.e., $||x_n - x||^* \to 0 \Longrightarrow ||x|| \leq \underline{\lim}_n ||x_n||$). He also investigated the form of γ-continuous linear functionals (i.e., those $f \in X'$ for which $x_n \overset{\gamma}{\to} x \Longrightarrow f(x_n) \to f(x)$) for some concrete two-norm spaces, for instance, $\ell^\infty_S = (\ell^\infty, ||\cdot||, ||\cdot||^*)$ and $L^2_S = (L^2, ||\cdot||, ||\cdot||^*)$ etc., where $||\cdot||$ denotes the usual norm of these spaces and $||\cdot||^*$ respectively stand for the functions $||x||^* = \sum_{n\geq 1} \frac{|x_n|}{2^n(1+|x_n|)}$ and $||f||^* = \int_0^1 |f(x)|dx$.

A systematic development of the theory of two-norm spaces came to limelight in the joint contribution of Alexiewicz and Semadeni when they published their work in the years 1958, 1959 and 1960. Indeed, motivated by the paper [141] due to Wiweger, they constructed in [14] a Hausdorff locally convex topology T on a two-norm space X_S such that the T-sequential convergence coincides with the γ-convergence and T-sequential dual of X is the same as the γ-dual X^*_γ which is the class of all γ-continuous linear functionals. Another significant result was to represent a γ-continuous linear functional as the uniform limit of a sequence of members of X^*_2 on the unit ball of

$(X, ||\cdot||)$; in other words, X_2^* is dense in X_γ^* relative to the dual norm topology of X_1^*. Using this, they deduced the general form of γ-continuous linear functionals for several concrete two-norm spaces including some known ones, for instance, $(C(-\infty,\infty), ||\cdot||, ||\cdot||^*)$, $(L^\infty(-\infty,\infty), ||\cdot||, ||\cdot||^*)$ where $C(-\infty,\infty)$ is the space of all continuous and bounded functions on $(-\infty,\infty)$ and $L^\infty(-\infty,\infty)$ is the space of all bounded and measurable functions on $(-\infty,\infty)$. The authors also successfully illustrated with an example that the analogue of Hahn-Banach theorem does not hold good for γ-continuous linear functional in general. However, they have shown in [13] that for two-norm spaces which are simultaneously vector lattices such an extension of γ-continuous linear functional is possible.

In the year 1959 and 1960, Alexiewicz and Semadeni [15,16] developed the structural properties of two-norm spaces on the lines of Banach space analogues. In their paper [15] they introduced the notions of γ-completion of a two-norm space and γ-conjugate, γ-separable, γ-precompact, γ-compact and γ-reflexive spaces for γ-convergence of sequences in a natural way and proved several results interrelating these notions. The two authors illustrated these notions with examples, for instance, we quote one such example of a γ-precompact space which is a Banach space $(X, ||\cdot||)$ with a Schauder basis $\{x_i; f_i\}$ where the second norm $||\cdot||^*$ is defined as

$$||x||^* = \sum_{n\geq 1} \frac{1}{2^n} |f_n(x)|.$$

They also proved

"A two-norm space γ-conjugate to a γ-reflexive space is γ-reflexive."

"Any normal γ-compact two-norm space is γ-reflexive."

"Any normal two-norm space can be embedded in a normal, γ-complete two norm space such that every γ-continuous linear functional f can be extended to its γ-completion with preservation of the first norm."

Relating the reflexivity of $(X,||\cdot||)$ with γ-reflexivity of $X_S = (X,||\cdot||,||\cdot||^*)$, the authors proved

"A normed space $(X,||\cdot||)$ is reflexive if and only if for any coarser norm $||\cdot||^*$ the two-norm space $X_S=(X,||\cdot||,||\cdot||^*)$ is γ-reflexive and saturated."

Towards the end of this paper, the authors characterized a saturated two-norm space (i.e., $X_\gamma^* = X_1^*$) in several equivalent forms. In the paper [16], the authors generalized the notion of a normal two-norm space to define a quasi-normal two-norm space which means that $||x_n-x||^* \to 0 \implies ||x|| \leq M \underline{\lim}_n ||x_n||$ for some constant M and showed that all the main properties of a normal two-norm space are preserved in this case. Introducing the notion of γ- semireflexivity they proved

"A two-norm space is γ-reflexive if and only if it is γ-semireflexive and quasi-normal."

Besides characterizing starred norm $||\cdot||^*$ and the possible spaces X_2^*, the authors derived sufficient conditions

for a given subspace X_1^* to be the γ-dual - a fact which was later found very useful by Subramanian [136] in his study of two-norm spaces and Schauder decompositions. In the last section of this paper, the authors established the following characterization of a reflexive Banach space as an application of their results of the earlier sections

"A Banach space $(X, ||\cdot||)$ is reflexive if and only if for every norm $||\cdot||^*$ coarser than $||\cdot||$, the dual of $(X, ||\cdot||^*)$ is dense in the dual X_1^* of $(X, ||\cdot||)$ relative to the dual norm topology of X_1^*."

After the publication of these three fundamental papers [14,15,16] Semadeni alone continued the study of two-norm spaces. In [118] he proved that every normal, γ-separable two-norm space $(X, ||\cdot||, ||\cdot||^*)$ can be embedded in the space of all bounded continuous real valued functions defined on the half line $[0,\infty)$ and equipped with norms $||f|| = \sup_{0\leq x<\infty} |f(x)|$ and $||f||^* = \sum_{n\geq 1} \frac{1}{2^n} \sup_{1\leq x\leq n} |f(x)|$, the embedding being isometric in $||\cdot||$ and isomorphic in $||\cdot||^*$. The author also dealt with the extension problem of γ-continuous linear functional from a γ-closed subspace to the whole in his papers [118,119] which was initiated by him in conjunction with Alexiewicz [14] around the year 1958. Whereas in [118] he proved that such an extension is not possible in general; but in [119] he showed that the restriction of γ-reflexivity on the subspace always yields the existence of such an extension. However, one may

find a categorical study of two-norm spaces in [120].

In 1964, Garling [43] noted that though there are several properties which are common between two-norm spaces and Fréchet spaces, the characterization of γ-compact sets by γ-convergent sequences is not one of them, for instance, he proved

"If $(X, ||\cdot||, ||\cdot||^*)$ is an infinite dimensional γ-compact two-norm space, and if $(X, ||\cdot||)$ is a reflexive Banach space, then there exists no γ-Cauchy sequence where γ-closed absolutely convex cover contains the γ-compact set B."

After a gap of several years Alexiewicz [3] revived his interest in the study of two-norm spaces and in 1974 replacing the coarser norm $||\cdot||^*$-topology by a locally convex topology μ weaker than the first norm $||\cdot||$-topology, he proved the existence of a mixed locally convex topology η on X such that the topological dual of (X,η) is the space of all linear functionals f on X for which $f|_B$ is continuous relative to $\mu|_B$ for each $||\cdot||$-bounded subset B of X. He also observed that his earlier study on two-norm spaces [14] follows as a particular case if μ is metrizable.

In the same year Alexiewicz [4], Alexiewicz and Golusda [9] concentrated on some known spaces equipped with two topologies in order to find the form of the γ-dual in each case. Indeed, Alexiewicz [4] considered the Hardy space H^1, the space of functions analytic in the open disc Δ, and showed that the γ-dual of this space is $\frac{C}{A_o}$, where the two topologies on H^1 are

the Hardy's norm topology and the compact open topology and C is the class of all continuous complex valued functions on $|z| = 1$ and A_o, the closed linear span in C of the exponential function $e^{in\theta}$, $n > 0$; whereas Alexiewicz and Golusda [9] dealt with the space H^∞, the space of bounded holomorphic functions on the unit disc Δ to determine the γ-dual and showed that the space H^∞ is γ-separable.

Around 1976 Alexiewicz [5] considered a vector space X equipped with two topologies defined with the help of a total pointwise bounded sequence $\{f_n\}$ on X. In fact he defined $||\cdot||$ and $||\cdot||^*$ as follows

$$||x|| = \sup_n |f_n(x)| \text{ and } ||x||^* = \sum_{n\geq 1} \frac{1}{2^n} |f_n(x)|.$$

Then he showed that the two-norm space $(X, ||\cdot||, ||\cdot||^*)$ is γ-complete if and only if it is γ-compact; and a γ-continuous linear functional f has the form

$$f(x) = \sum_{n\geq 1} \alpha_n f_n(x), \; x \in X$$

for some $\{\alpha_n\} \in \ell^1$. As a particular case of this result he deduced an earlier contribution on H^∞ mentioned above. On the lines of his contribution in [5], Alexiewicz [6] considered the two-norm structure on bv, the first and the second norm being given by

$$||\alpha|| = |\alpha_o| + \sum_{n\geq 1} |\alpha_n - \alpha_{n+1}|$$

and

$$||\alpha||^* = \sup_{n \geq 0} |\alpha_n| .$$

He determined the γ-dual of this space and established multiplicative property of these norms corresponding to co-ordinatewise multiplication of sequences. He also proved the γ-continuity of the multiplicative inversion. Further contributions of Alexiewicz to the study of two-norm algebras are contained in his papers [7,8]. In [7] he investigated conditions under which separate continuity of multiplication yields joint continuity and dealt with the completion of two-norm algebras, continuity of multiplication in Wiweger topology, continuity of inversion and the representation of two-norm algebras; whereas in [8], he gave an address at the Fourth Prague Topological Symposia, 1976, briefly discussing topics related to the continuity of multiplication and inversion, sets of continuous characters and Gel'fand representation.

2. Mixed Topology:

Having known the notion of mixed convergence, namely, γ-convergence of sequences in a vector space X equipped with two norm topologies, it was a natural anxiety for mathematicians to find a topology on X that gives rise to a sequential convergence which coincides with the γ-convergence of sequences. In fact, Alexiewicz [2] proved around 1954 that if γ-convergence is metrical then it is nothing but the norm convergence induced by the first norm. So the researchers were interested in finding

a nonmetrizable locally convex topology on X, for which the sequential convergence coincides with the γ-convergence. The first attempt in this direction was made by A. Wiweger [141] around the year 1957, who extended the theory of two-norm space to the most general setting of linear topological spaces. In fact, he considered a linear space X equipped with two linear topologies T_1 and T_2 and showed that the topology $\tau = \gamma[T_1, T_2]$ generated by the collection of sets of the type $\bigcup_{n=1}^{\infty} \sum_{i=1}^{n} (v_i \cap iu)$, where $\{v_n\}$ is a sequence in $\mathcal{U}_{T_2}$ and u is in $\mathcal{U}_{T_1}$ is the required mixed topology for which γ-convergence of sequences relative to T_1 and T_2 coincides with the τ-sequential convergence. Further he showed that τ is the strongest topology such that τ and T_2 coincide on T_1-bounded subsets of X and if T_1 is a norm topology then τ is the weakest topology for which a linear mapping f from X to a TVS (Y,T) is τ-T continuous if and only if $f|_B$ is $T_2|_B$-T continuous for each T_1-bounded set B in X.

After a couple of years, Wiweger in his doctoral dissertation [143], published a number of results on the mixed topology including his earlier contributions in [141]. The important features of this thesis [143] are the validity of the analogue of Hahn Banach theorem for γ-continuous linear functional under suitable restrictions and the precise representation of seminorms generating the mixed topology. Indeed, Wiweger discovered that for the extension of γ-continuous linear functional from a subspace to the whole space, the

intrinsic property used in the condition

$$\gamma[T_1, T_2]|_{X_o} = \gamma[T_1|_{X_o}, T_2|_{X_o}]$$

satisfied on a subspace X_o of X, which is not true in general. He did overcome this difficulty by putting suitable restriction on the unit ball of $(X, ||\cdot||)$ and proved

"Let $(X, ||\cdot||, ||\cdot||^*)$ be a two-norm space such that $(X, ||\cdot||^*)$ is a F-space and the unit ball $S = \{x : ||x|| \leq 1\}$ is T_2-compact. Then γ-continuous linear functional defined on a T_2-closed subspace of X can be extended to the whole space X." Wiweger concluded this paper with several examples of concrete mixed topologies.

In his later contribution [142], Wiweger observed that Semadeni's result (cf. [119] and also the preceding section) which generalizes the above extension theorem and was proved by a different method, can be deduced partly by the method applied in [143]. Instead of the γ-reflexivity of the subspace Wiweger assumed the γ-reflexivity of the whole space and proved it in the following form

"Let $(X, ||\cdot||, ||\cdot||^*)$ be a γ-reflexive two-norm space and X_o be a closed subspace of X relative to the mixed topology $\gamma(||\cdot||, ||\cdot||^*)$. Then a γ-continuous linear functional defined on X_o can be extended to the whole space X."

In the same paper he also proved the analogue of the Šmulian and Eberlin theorem in the following form

"Let $(X, ||\cdot||, ||\cdot||^*)$ be a quasinormal γ-complete two-norm space and A be a subset of X. Then the following statements are equivalent

(i) A is $\sigma(X, X_\gamma^*)$-sequentially compact;

(ii) A is $\sigma(X, X_\gamma^*)$-relatively countably compact;

and (iii) $\sigma(X, X_\gamma^*)$-closure of A is compact."

Around the year 1963, Persson [95] defined the notion of a mixed topology T on a vector space X equipped with two locally convex topologies T_1 and T_2 such that every T_2-bounded set is T_1-bounded, in such a fashion that T coincides with Wiweger's topology under special circumstances and the theory of locally convex spaces is applicable in order to obtain results which generalize and sharpen the known results of two-norm spaces; for instance, the results on the relationships among the first, second and γ-duals and also on the γ-reflexivity of the space. He termed a triplet (X, T_1, T_2) where T_1 and T_2 are defined as above a bi-topological space and defined the mixed topology T as the finest locally convex topology which coincides with T_1 on T_2-bounded subsets. Besides establishing several results on the mixed topology, he characterized the reflexivity of a bornological space, thus generalizing an earlier result of Alexiewicz and Semadeni [16] quoted in the preceding section. Concerning extension of γ-continuous linear functional, he provec

"If a bi-topological space (X, T_1, T_2) is b-normal (i.e., (X, T_2) has a fundamental system of countable bounded sets which

are absolutely convex and T_1-closed) or C-normal (i.e., (X,T_2) has a fundamental neighbourhood system of absolutely convex and T_1-closed sets at origin) and (X,T) is reflexive, then every T-closed subspace Y of X has the extension property for γ-continuous linear functionals."

A year later, Garling [42] concentrated on the topological principles upon which the properties of two-norm spaces were based and in the process he discovered the use of the generalized inductive limit topologies and the related results in his study of pseudo two-norm spaces - a class which is slightly more general than the class of two-norm spaces. Indeed, a metrizable l.c. TVS (X,T_2) equipped with an increasing sequence $\{B_n\}$ of absolutely convex, T_2-closed and bounded sets is said to be a two-norm space provided (a) $\bigcup_{n=1}^{\infty} B_n = X$ and (b) for each i and $a > 0$ there exists a j such that $aB_i \subset B_j$. If (b) is replaced by (b′) $B_i = iB_1$, then X is called a two-norm space. According to his terminology, he defined the γ-convergence of a sequence as follows :

$x_n \xrightarrow{\gamma} x_o$ if $\{x_n\} \subset B_i$ for some i and $x_n \xrightarrow{T_2} x_o$. He applied the theory of generalized inductive limit topology to pseudo two-norm spaces; for instance, using Grothendick's completion theorem in terms of generalized inductive limit topology, he proved

"If $X^* = X_2^*$, then T_2 is a normable topology, $\{B_i\}$ is a fundamental sequence of T_2-bounded sets and $T_2 = T$."

He also considered duality properties of a pseudo two-norm space and observed that X^* is a Fréchet space with respect to the topology θ_1 of uniform convergence on B_i's and X_2^* is a dense subspace of X^*. He also proved that the T-completion of X in X_2^{**}, the dual of X_2^* relative to θ_1, is the closure of X with respect to pseudo two-norm topology on X_2^*. He also discussed the γ-compactness, γ-precompactness and γ-reflexivity in terms of the corresponding notions relative to the topology T.

Around the year 1965, Arima and Orihara [18] came out with a general method forming the neighbourhood system for generalized mixed topologies which includes the mixed topologies of Wiweger and Persson as particular cases. Indeed, they defined the general mixed topology with the help of a locally convex topology T on X and a family $\mathcal{U}$ of subsets of X satisfying the properties : (i) for $A \in \mathcal{U}$, $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $\alpha A \in \mathcal{U}$; (ii) For $A \in \mathcal{U}$, $\alpha \in \mathbb{R}$ with $|\alpha| \leq 1$, $\alpha A \subset A$; and (iii) for $x \in X$, there exists a A in $\mathcal{U}$ such that $x \in A$, which he termed as primitive. Also for a linear space X equipped with two locally convex topologies T_1 and T_2, he showed that the generalized mixed topology is the finest locally convex topology which is identical with T_2 on members of $\mathcal{U}$ where $\mathcal{U}$ may any of the collections (i) all T_1-bounded and absolutely convex subsets of X; (ii) all T_1-totally bounded and absolutely convex subsets of X; (iii) all T_1-compact and absolutely convex subsets of X; (iv) all T_1-equicontinuous absolutely convex subsets of X; (v) a basis of

neighbourhood at origin in the T_1-topology. If $\mathcal{U}$ is the class (i), the generalized mixed topology coincides with Persson's mixed topology in case of an l.c. TVS and with Wiweger's mixed topology in case of two-norm spaces.

A year later in 1966, Orima [17] alone brought out a paper on generalized mixed topology wherein he proved several properties of generalized mixed topology and gave a suitable form of a neighbourhood at the origin for the mixed topology in the dual space. He also mentioned several examples of mixed topologies on dual spaces and investigated properties of some of them.

Some generalizations of the mixed topology were also discussed by Precupanu [100] in the year 1967.

J.B. Cooper [29] in 1971 showed that the strict topology on spaces of continuous functions introduced by R.C. Buck [20], is a particular case of the mixed topology due to Wiweger. Besides, the author also pointed out that the results on strict topology could be derived in a simple way via the tools of mixed topology. In his later work [30], Cooper proved that the mixed topologies on some vector spaces coincide with the Mackey topologies. A detailed account of his contribution as well as of others in the direction of Saks spaces, mixed topologies via a bornology, strict topologies etc. is to be found in his recent monograph [31].

In the year 1972, Roelcke [105] considered a sequence of absolutely convex sets $\{A_n\}$ in an l.c. TVS (X,T) such that $\bigcup_{n \geq 1} A_n$ is absorbing in X and investigated several properties of the finest locally convex topology $\tilde{T}$ which coincides with T on each A_n, $n \geq 1$. Apart from giving the precise form of the filter base generating the neighbourhood system at origin for the topology $\tilde{T}$, the author simplified an earlier characterization of a $\tilde{T}$-bounded set in X due to Garling (cf. [42], Theorem 2(2), p.8) namely, a T-bounded subset B is $\tilde{T}$-bounded if and only if B is absorbed by T-closure of A_n for some n and added to the theory of pseudo two-norm spaces of Garling [42], especially by considering the precompact or compact sets A_n, $n \geq 1$. Further, he made a comparative study of the topology $\tilde{T}$ with that of Wiweger's mixed topology [143] and sharpened some of his results in the light of the topology $\tilde{T}$.

Around the year 1973, B. Perrot [92] analysed the construction of mixed topology in view of the notion of convex bornology (cf. [56]) [a convex bornology is a family $\mathcal{B}$ of absolutely convex subsets of a linear space X which does not contain any non-trivial subspace satisfying (i) $X = \bigcup B$, $B \in \mathcal{B}$; (ii) if $B \in \mathcal{B}$, $\lambda > 0$, then $\lambda B \in \mathcal{B}$; (iii) $B, C \in \mathcal{B} \Longrightarrow$ there exist $D \in \mathcal{B}$ such that $B \cup C \subset D$; and (iv) $B \in \mathcal{B}$ and $C \subset B \Longrightarrow C \in \mathcal{B}$, where C is an absolutely convex set not containing a non-trivial subspace] and extended the principal results in the theory of mixed topology to the case when the underlying

linear topology is not necessarily a convex one. The novelty of his results lies essentially in the proofs that are not easily established in the case of linear topology by using the techniques of duality theory. Perrot [93] continued his study of mixed spaces, namely of a triplet $(X,T,\mathcal{B})$ consisting of a vector space X, a locally convex topology T and a convex bornology $\mathcal{B}$ of T-bounded sets, in [93] and [94]. He defined the notion of γ-convergence in this mixed space as follows :

"A sequence $\{x_n\}$ is said to be γ-convergent to a point x if $\{x_n\} \subset B_i$ for some B_i of the bornology $\mathcal{B}$ and $x_n \xrightarrow{T} x$."

Using the concepts of duality theory he showed that $(X^*,\mathcal{B}^o,T^o)$ is also a mixed space where $\mathcal{B}^o$ is the S-topology on $\mathcal{B}$ and T^o is the bornology of the polars of the T-neighbourhood and introduced notions like γ-semireflexivity (that is, $(X^*,\mathcal{B}^o)^* = X$) and γ-reflexivity (that is, $(X,T,\mathcal{B}) = (X,T,\mathcal{B}^{oo})$) which he characterized using Bi-polar theorem. In [94] he gave the precise form of a member of the neighbourhood system at origin for the linear topology $\tilde{T}$ generated by the γ-closed sets (a set which contains all γ-limits of its elements). The author established connections of the topology $\tilde{T}$ with the topologies T^ℓ, T^V and T^g which are respectively the finest locally convex, vector and general topology on X coinciding with T on the members of $\mathcal{B}$. Answering a question posed by Garling ([42],p.23) whether there exists a two-norm space for which T^g is strictly finer than T^ℓ, the author gave an example of a two-norm space for which T^ℓ is strictly coarser than T^g.

In 1974 Serafin [122] defined a Hausdorff locally convex topology on a normed space $(X, ||\cdot||)$ with the help of a total subset A of X^*; indeed, generated by the basis of neighbourhoods at origin of the form

$$\{x \in X : |\langle x, f_n \rangle| < \alpha_n, \forall n \geq 1\},$$

for all $\{f_n\}$ in A and strictly increasing $\{\alpha_n\}$ in $\mathbb{R}_+$ and showed that it coincides with Wiweger's topology when A is countable; and in this case the precise representation of continuous linear functionals has also been formulated.

Lastly it would be worthwhile to name Stroyan [134], Cook and Dazord [26], Noureddine [80, 81, 82, 83, 84] for their related contributions. Indeed, Stroyan [134] used non standard analysis for his study of mixed topology; Cook and Dazord [26] considered the notion of γ-convergence for filters in a set equipped with a convergence structure; whereas Noureddine [80,81,82,83,84] studied mixed structure in view of limit spaces.

3. Sequence Spaces and Schauder Decompositions :

This section has been divided into two subsections.

Sequence spaces : The theory of scalar-valued sequence spaces (SVSS) which indeed occupies a priviledged position because of its varied applications in the theory of Schauder bases [37, 61,74,130], structural study of locally convex spaces [60,69], nuclear operators and spaces [99,133], summability domains [19],

matrix transformations [51,91] and the theory of bi-l.c.TVS (cf. Chapter 3 and Chapter 4) was put forward in the systematic form by Köthe and Toeplitz [70] around the year 1934; and in particular has been proved to be very rich in providing counter examples for various concepts, results in all the branches mentioned above. For a systematic account of the development of this rich theory, we refer to the doctoral dissertations of Patterson [91] and Rao [101]; and the monographs [60], [61] for its literature.

A generalization of the Köthe theory of sequence spaces in the form of VVSS appeared in the monograph [98] of Piétsch around the year 1962, who introduced a VVSS $\lambda(X)$ corresponding to a perfect SVSS λ and an l.c. TVS (X,T) as follows

$$\lambda(X) = \{\{x_i\} : x_i \in X,\ i \geq 1 \text{ and } \{f(x_i)\} \in \lambda\ ,\ \forall\ f \in X^*\}$$

and topologized the same with the help of topologies on X and λ. He also studied the intrinsic properties of sets like boundedness, compactness etc. in $\lambda(X)$ and dealt with the representation problem of linear operators. In 1967, Phoung Các [97] made his contribution in the theory of VVSS on the lines of Köthe and Teoplitz [70] and obtained several results for the dual pair $<\Lambda(X), \Lambda^{x}(Y)>$. Gregory [45] carried out the study of VVSS in his thesis and generalized some of Các's results. He introduced a solid topology on $\Lambda(X)$ which envelopes the normal topology, and studied the space $\lambda(X)$ defined in Section 4 of Chapter 1.

In 1970, De Grande-De Kimpe [33] and Rosier [106] broughtout their theses on the study of the space $\lambda\{X\}$. Though there is a overlapping of some results in these two dissertations yet the approaches are basically different. Whereas Rosier [106] made a comparative study of the spaces $\lambda[X]$ and $\lambda\{X\}$ and attempted to unify these two approaches of study; De Grande-De Kimpe developed the structural study of $\lambda\{X\}$ in order to use the same for her results on λ-nuclear operators, operators of λ-type (cf. [34])

Towards the end of seventies Gupta, Kamthan and Rao [48,50, 53,54] carried out further investigations on VVSS $\Lambda(X)$ and discovered the form of the generalized Köthe duals of several known VVSS, e.g. $\ell^1(X)$, $m_o(X)$, $c_o(X)$, $c(X)$, $\ell^\infty(X)$ etc. In [54] the trio discovered that every VVSS $\Lambda(X)$ has a $\sigma(\Lambda(X), \Lambda^\times(X^*))$ Schauder decomposition. They exploited this fact to relate the structure of $\Lambda(X)$ with the types of this S.D. and proved several results on VVSS $\Lambda(X)$ for an l.c. TVS X with an S.D. or without an S.D. The study of VVSS was further continued by Gupta, Kamthan and Patterson when they tackled the problem of representing the topological dual of a VVSS as a VVSS in [52]. Also, Gupta and Patterson [51] dealt with the problem of representing linear transformations on VVSS as matrices of operators on the underlying spaces for the first time. A heuristic and uptodate development of this theory is to be found in [91].

<u>Schauder decompositions</u> : The notion of an S.D. may be traced back to 1949 in the work of M.M. Grinblyum [46] and its study

has importance in those spaces which do not possess a Schauder base; for now it is a known fact that there exists a separable Banach space having no Schauder base (cf. [38]). Indeed, the theory of S.D. has been found useful in the nuclear theory of locally convex spaces [99,133]; study of two-norm spaces [135,136,137] and also the VVSS [54]. A detailed account of its development is given in [102]; however, we mention a bird's eye view of the same in this subsection.

After Fage's study [39] of S.D. in Hilbert spaces around 1950, Sanders introduced the notion of S.D. in a Banach space and observed in [116] that every Banach space possesses a decomposition not necessarily a S.D. He also generalized in [115] the well known result of James concerning the characterization of the reflexivity of a Banach space in terms of its Schauder basis to Banach spaces having S.D.

Fascinated by the importance of S.D., McArthur and Retherford jointly as well as individually contributed substantially to this theory in Banach spaces as well as in general TVS during 1962 to 1966, and established several results concerning the characterization of Schauder decomposition, T-uniform, shrinking S.D. (cf. [79]) and stability of S.D. etc. They introduced the notion of e-S.D. in TVS; studied the notion of S.D. in a complete linear metric space; characterized a totally bounded and compact subset of a TVS with the help of S.D. in [79]

and established the weak basis theorem (i.e., A weak-S.D. for a barrelled space (X,T) is T-S.D.) in [78].

W.H. Ruckle [108,109], who introduced the notion of a boundedly complete S.D. in a Banach space gave a shorter proof of Sander's result on reflexivity and in [108] proved

"Let $\{M_n;P_n\}$ be an S.D. for a Banach space $(X,||\cdot||)$. If each M_n is reflexive and $\{M_n\}$ is boundedly complete then X is topologically isomorphic to the dual space of $\sum_{n\geq 1} P_n^*(X^*)$."

Inspired by a result of Sanders [116], D.W. Dean [36] in 1966 established that the space ℓ^∞ does not contain S.D. at all and he categorized in general the Banach spaces without S.D.

In 1965, Russo presented a systematic study of e-S.D. and monotone S.D. in his dissertation [111] (also in [112]).

Around 1968, D.J. Fleming [41] extended the idea of S.D. to introduce the notion of generalized S.D. in a TVS and proved results of McArthur, Retherford etc. for generalized S.D.

In the direction of generalizing the results of S.D. in Banach spaces to the general setting of locally convex spaces, T.A. Cook [27,28] was the first to prove the generalized version of Sander's result on reflexivity which runs as follows :

"If X is semi-reflexive l.c. TVS with an S.D. $\{R(P_n);P_n\}$ then the S.D. is both shrinking and boundedly complete; and conversely if X is an l.c. TVS with boundedly complete and shrinking S.D. $\{R(P_n);P_n\}$ such that each $R(P_n)$ is semireflexive then X is semireflexive."

N.J. Kalton who correlated e-S.D. with certain topological properties of an l.c. TVS in [59], introduced the notion of a simple S.D. in an l.c. TVS in [58] and characterized semi-reflexivity of the space in terms of several equivalent conditions involving simple and boundedly complete character of S.D.

Around 1972, Kamthan and his group took up the study of S.D. in general TVS and established several results on characterization of S.D. in X^* [63]; equivalence of Schauder decompositions [64]; Schauder decompositions in dual and bidual spaces [62] and characterization of shrinking and uniform S.D. [65].

Finally, we also mention the names of Chadwick [21,22] and De Wilde [35] for their important contributions in the theory of S.D.

4. <u>Applications</u> : As mentioned in the beginning of this chapter, the theory of two-norm spaces has been found useful in the theory of summability as well as in the study of S.D. in Banach spaces. We shall briefly touch upon these two aspects one by one in the following few-pages.

The applications of two-norm spaces to the summability theory are to be found in [10,11,12,88,96,121,131,132,139]. To begin with, Orlicz [88] applied the theory of Saks spaces to the theory of linear methods using the terminology of [77] in the year 1957. Later in [10,11] Alexiewicz and Orlicz dealt with the extension of single sequence summability to double sequence summability and applying two-norm space theory establish

results involving the linear methods operating on the class of double sequences. In [12] the authors established the two-norm structural properties like γ-completeness and γ-dense property of the space ϕ in certain domains of convergence which are nothing but the sequence spaces $\ell^{\infty} \cap c_o^m$, where c_o^m stands for the class of sequences m - A summable to zero [Given two Banach spaces X and Y and a system $A = \{A_{ij}\}$ of linear operators from X to Y, if $A_i(\{\alpha_j\}) = \sum_{j\geq 0} A_{ij}(\alpha_j)$ convergent for $\bar{\alpha} \in X$ and for each $i \geq 1$ and if $\lim_{i\to\infty} A_i(\bar{\alpha}) = A(\bar{\alpha})$ (resp. $\omega\text{-}\lim_{i\to\infty} A_i(\bar{\alpha}) = A(\bar{\alpha})$) then $\bar{\alpha}$ is said to be strongly A-summable (resp. weakly-A-summable) to $A(\bar{\alpha})$. In case, the series representing $A_i(\bar{\alpha})$ are weakly convergent and $A_i(\bar{\alpha})$ converges strongly to $A(\bar{\alpha})$ then $\bar{\alpha}$ is said to be m-A-summable to $A(\bar{\alpha})$.]

Motivated by the terminology of summability methods like conull, coregular FK-spaces [an FK-space X is called conull if $e^{(n)} \to e$ weakly; otherwise coregular] Synder [131] studied these spaces and using the tools of two-norm space theory, he generalized a well known summability inclusion theorem of Zeller [145]. Indeed, he proved

"Let $(\lambda, ||\cdot||_2)$ be a conservative FK-space [i.e., $c \subset \lambda$]. Then the following are equivalent

(i) $W(r) \subset X$ for some r;

(ii) X is conull;

(iii) $e \in \gamma$-linear closure of $\{e^n\}$ in the two-norm space

$(c, ||\cdot||_\infty, ||\cdot||_2)$ [In short, it is written as $e \sim E^\infty(c)$]; where $W_r = \{\{\alpha_i\} : ||\bar{\alpha}||_{r,n} \to 0\}$, $r = \{r_i\}$ is a strictly increasing sequence of integers with $r_1 = 1$ and $||\bar{\alpha}||_{r,n} = \max\{|\alpha_n - \alpha_m| : r_n \leq m < n \leq r_{n+1}\}$. Indeed, the equivalence (ii) <==> (iii) in Synder's theorem, characterizes the conull FK-spaces containing all convergent sequences as those FK-spaces for which the constant sequences are "close to" the members of ϕ in an appropriate mixed topology. Sember [121] generalized a part of this characterization for a broader class of FK-spaces known as variational FK-spaces which are defined as FK-spaces containing bv, the space of all sequences of bounded variation with norm

$$||\{\alpha_n\}||_{bv} = |\alpha_1| + \sum_{k=2}^{\infty} |\alpha_k - \alpha_{k-1}|$$ and proved

"$(\lambda, ||\cdot||_2)$ be a variational FK-space. If λ is conull then $e \sim E^\infty(bv)$ relative to γ-convergence of $(bv, ||\cdot||_{bv}, ||\cdot||_2)$."

He also investigated conditions under which the γ-convergence implies the conullity of a variational FK-space and applied these results to infer the weak compactness or compactness of matrix operators transforming bv to c; and bv to bv respectively.

Considering the Saks space formed by the unit ball of $(\ell^\infty, ||\cdot||_\infty)$ and the metric induced by $||\{\alpha_n\}||_2 = \sum_{n\geq 1} \frac{|\alpha_n|}{2^n}$, Conway [25] gave a simplified proof of the Schur's theorem which states that the weak sequential convergence in ℓ^1 implies the norm convergence.

Around 1972, Synder and Wilansky [132] unified several known inclusion theorems for FK-spaces and using two-norm convergence criterion they proved a result analogous to Synder's characterization of a conull conservative FK-space for a semi-conservative FK-spaces [i.e., those FK-spaces λ for which $\sum_{n\geq 1} f(e^{(n)})$ is convergent for all $f \in \lambda^*$] in the following form.

"Let λ be semiconservative. Then λ is conull if and only if $e \in \lambda$ and there exists a sequence $\{\bar{\alpha}^i\}$ in λ such that $\bar{\alpha}^i \xrightarrow{\gamma} e$ relative to γ-convergence of the two-norm space λ with second norm defined by

$$||\{\alpha_i\}||_{bv} = \sum_{n\geq 1} |\alpha_n - \alpha_{n+1}| + |\lim_i \alpha_i|."$$

They also mentioned examples in support of the fact that semi-conservative character can not be omitted in the preceding characterization.

Bennet and Kalton [19] around 1972 made a detailed study of FK-spaces containing c_o. They improved an earlier result of Synder [131] on conull FK-spaces and identified the mixed topology on $W \cap \ell^\infty$, where for an FK-space λ containing c_o, $W = \{\bar{\alpha} \in \lambda : \bar{\alpha}^{(n)} \to \alpha \text{ weakly}\}$, with the Mackey topology $\tau(W \cap \ell^\infty, \ell^1)$ with respect to which they showed the completeness of the space. They applied all these results to give an example of an l.c. TVS with a weak Schauder basis which is not a Schauder basis and to obtain a result, namely - "An arbitrary FK-space λ with $c_o \cap \lambda$ not closed in λ, contains a bounded divergent

sequence" which was used by Meyer-König and Zeller in summability theory.

In 1973, Chilana [23] showed that the space ℓ^∞ equipped with the Wiweger's mixed topology has some interesting properties, e.g., every closed linear subspace of ℓ^∞ is a closed linear span of a bounded subset and the sequence $\{e^n\}$ is an unconditional, monotone, boundedly complete, e-Schauder basis for ℓ^∞ and ℓ^∞ equipped with the mixed topology is separable, contrary to its nature of inseparability under the sup norm topology.

As mention earlier, the theory of two-norm spaces has applications in the summability theory as well as in the Schauder basis theory, but it also plays a very significant role in the study of Schauder decompositions. Indeed, the importance of two-norm spaces or Saks spaces theory was unfolded by Subramanian and Rothman in their doctoral dissertations in early seventies. Whereas Subramanian applied the theory of two-norm spaces to the study of S.D., Rothman showed how the theory of two-norm spaces (and Saks spaces) could be used to tackle problems on Banach spaces. However, to be in tune with our contributions in this thesis, we shall only confine our attention to the applications of two-norm spaces in S.D.

Indeed, in his paper [135] which forms a part of his thesis, Subramanian introduced the notion of a canonical two-norm space X_S for a Banach space X with a S.D. $\{M_n;P_n\}$ and related the mixed structure of the space with the types of its S.D. as follows

"Let $\{M_n; P_n\}$ be a S.D. for a Banach space $(X, ||\cdot||_1)$ and X_S be the corresponding canonical two-norm space. Then $\{M_n; P_n\}$ is boundedly complete (shrinking) if and only if X_S is γ-complete (saturated)."

Further he proved that $\{P_k^*(X_1^*); P_k^*\}$ is a S.D. for the γ-dual X_γ^* of X_S in the dual norm of $||\cdot||_1^*$ of $X_1^* = (X, ||\cdot||_1)^*$ which he exploited to introduce the notion of k-reflexivity related with the S.D. in the following form

"X_S is called k-reflexive if the canonical embedding $J : X \to X_{\gamma\gamma}^{**}$, defined by $(Jx)(f) = f(x)$ for all $f \in X_\gamma^*$ is onto, where $X_{\gamma\gamma}^{**}$ is the γ-dual of the canonical two-norm space of $(X_\gamma^*, ||\cdot||_1^*)$."

The relationship of γ-reflexivity with that of k-reflexivity is exhibited in [135,137] and runs as follows :

"Let $(X, ||\cdot||_1)$ be a Banach space with S.D. $\{M_n; P_n\}$ then X_S is γ-reflexive if and only if it is normal, γ-complete and k-reflexive."

Applying his results on canonical two-norm spaces, Subramanian derived several known results in S.D. and the basis theory of Banach spaces; for instance, the famous result concerning the characterization of reflexivity of a space in terms of its S.D. by Sanders and Ruckl's result on isomorphic character of a Banach space X which we have already mentioned in the preceding section. He also dealt with the relationship of the decompositio

constant with the constant of quasinormality and proved a result more general than that of Singer ([129], p. 126) concerning the constant of basic sequences in Banach space for the setting of S.D.

In 1973, Subramanian and Rothman [137] considered the γ-completion of a canonical two-norm space and discussed the Y-pseudo reflexivity and Y-reflexivity of a Banach space X – a notion introduced by Singer [123,125] for a closed subspace Y of X^* [A Banach space X is called a Y-pseudo-reflexive if the canonical map J from X into Y^* is a topological isomorphism onto Y^*. If J is an isometry as well, X is called Y-reflexive]. Applying two-norm space theory, the authors presented a different proof of the following result of Singer [125, p.142].

"Let $(X, ||\cdot||_1)$ be a Banach space and Y a separable linear subspace of X_1^*. Then X is Y-reflexive if and only if S, the closed unit ball of $(X, ||\cdot||_1)$ is sequentially complete for the topology $\sigma(X,Y)$ and Y is an X-total subspace X_1^*."

They also gave several equivalent conditions for X_γ^*-pseudo reflexivity (resp. X_γ^*-reflexivity) of a Banach space with S.D. and studied the two-norm structure of direct sum of Banach spaces, which in a natural fashion, contains a generalized S.D.

Subramanian [136] continued his interest in this direction and in 1977 he came out with the generalization of some results in terms of the quasi-γ-reflexivity, quasi-k-reflexivity, r-boundedly complete and p-shrinking S.D. According to him – a

two-norm space X_s is called quasi-saturated of order n if X_γ^* has finite co-dimension n in X_1^* and quasi-γ-reflexive of order n if it is quasi-normal and J(X) has finite co-dimension n in $X_{\gamma 1}^{**}$. In terms of these two notions he gave several equivalent conditions for quasi-reflexivity of a Banach space - a notion introduced by Civin and Yood [24] and means the finite codimension of J(X) in X^{**}. For a Banach space with S.D. $\{M_k;P_k\}$ he proved

"If $(X,||\cdot||_1)$ is a Banach space with S.D. $\{M_k;P_k\}$, then X is quasi-reflexive of order n if and only if there exist non negative integers s, p and r (all $\leq$ n and uniquely determined by $\{M_k\}$) such that

(i) each M_j is quasi-reflexive of order m_j such that $\sum_{j\geq 1} m_j = s$; (ii) $\{M_k\}$ is r-boundedly complete (that is, $\dim \frac{C(X)}{J(X)} = r < \infty$); (iii) $\{M_k\}$ is p-shrinking (that is, X_s is quasi-saturated of order p); (iv) p+s+r = n."

The above result generalizes known results of Sanders ([115],p.205) and Singer ([128], Cor. 1) and if dim $M_k = 1$ for all $k \geq 1$ and n = 1, it reduces to a well known result of Cuttle ([32], Theorem 3.6). He also established the duality relationship of k-boundedly complete and k-shrinking decompositions with the help of two-norm space theory which is already known for bases due to Singer [128].

Fascinated by the several applications of two-norm space theory as outlined above and at the same time looking at the

absence of γ-convergence of a net in the literature of a general bi-topological space, Gupta and the author decided to take up this task and studied the same in a vector space equipped with two locally convex topologies one being finer than the other so as to introduce the notion of a bi-l.c. TVS. In the process, they discovered that the γ-convergence defined for nets is not a linear one (indeed, the scalar multiplication $(\alpha, x) \to \alpha x$ is not necessarily jointly continuous since a convergent net is not necessarily bounded) and so it is futile to imagine the existence of a linear topology having the same convergence as that of γ-convergence. However, turning over the pages of this thesis, one would find that the theory of bi-l.c. TVS has been developed to a certain extent and has also been applied to the study of S.D. in an l.c. TVS (X,T); yet this study is by no means complete, for instance, one may ask for the relationship of γ-convergence in an l.c. TVS with a Schauder basis and the natural order convergence defined by the basis, and also the generalizations of various results of Subramanian and Rothman for S.D.in an l.c. TVS. To discover the neighbourhood system for the topology which induces the same convergence as the γ-convergence is another task ahead. However, we believe, as the wheel of time rolls by, to have the solutions of these several unexplored problems and many others likewise on this beautiful and interesting theory of bi-l.c. TVS initiated in this thesis.

CHAPTER - 3

ELEMENTS OF BI-LOCALLY CONVEX SPACES

CONTENTS

1. INTRODUCTION :

This chapter is concerned with the development of the elementary structural properties of a bi-locally convex space which means a vector space X equipped with a pair of Hausdorff locally convex topologies, one being finer than the other. Whereas in Section 2, we introduce several notions on a bi-locally convex space so as to be able to use the same in the subsequent work of this thesis and illustrate some of these with examples; in the third section we concentrate on the study of normal and quasinormal spaces. Having characterized γ-bounded and γ-compact subsets of a bi-locally convex space, we introduce the concept of γ-γ-isomorphisms between two bi-locally convex spaces and establish their relationships with the isomorphisms relative to the original topologies in the fourth section. The final section of this chapter incorporates results related to the three duals, namely, the topological duals relative to the two topologies and the γ-dual of a bi-locally convex space.

2. Basic Concepts :

In this section we introduce several fundamental concepts which are needed throughout the sequel. We begin with

Definition 2.1 : (i) A bi-locally convex space (abbreviated hereafter bi-l.c. TVS) is a triplet (X, T_1, T_2) where T_1 and T_2 are Hausdorff locally convex topologies on X such that the topology T_1 is finer than T_2 and in short, we write X_b for

(X,T_1,T_2), that is, $X_b \equiv (X,T_1,T_2)$ and (ii) a bi-l.c. TVS X_b is said to satisfy the property (m) if there exists a fundamental system of balanced, convex, T_1-bounded and T_2-closed sets for the family of all T_1-bounded sets in X.

Remark : The bi-l.c. TVS $(\ell^1, ||\cdot||_1, \sigma(\ell^1, c_o))$ where $||\cdot||_1$ is the usual norm topology of ℓ^1, satisfies the property (m).

Definition 2.2 : In a bi-l.c. TVS $X_b \equiv (X,T_1,T_2)$, a net $\{x_\delta : \delta \in \Lambda\}$ in X is said to be (i) γ-convergent to a point $x \in X$, written as $x_\delta \overset{\gamma}{\to} x$ provided $\{x_\delta\}$ is T_1-bounded and converges to x relative to the topology T_2; and (ii) γ-Cauchy provided $\{x_\delta\}$ is T_1-bounded and T_2-Cauchy. A bi-l.c. TVS X_b is said to be (iii) γ-complete (resp. γ-sequentially complete) if every γ-Cauchy net (resp. γ-Cauchy sequence) in X_b, γ-converges to a point of X; (iv) quasinormal (resp. quasi-pseudonormal) if there exists a family $\mathcal{D}_1$ of seminorms generating T_1 such that for each $p \in \mathcal{D}_1$ there exists $q \in \mathcal{D}_1$ satisfying

$$x_\delta \to x \text{ relative to } T_2 \implies p(x) \leq \underline{\lim}_{\delta}\, q(x_\delta)$$

(resp. $x_\delta \overset{\gamma}{\to} x \implies p(x) \leq \underline{\lim}_{\delta}\, q(x_\delta)$); and (v) normal (resp. pseudonormal) if for each $p \in \mathcal{D}_1$

$$x_\delta \to x \text{ relative to } T_2 \implies p(x) \leq \underline{\lim}_{\delta}\, p(x_\delta)$$

(resp. $x_\delta \overset{\gamma}{\to} x \implies p(x) \leq \underline{\lim}_{\delta}\, p(x_\delta)$) where $\{x_\delta\}$ is a net in X, x a point in X and $\underline{\lim}_{\delta}\, p(x_\delta) = \sup_{\alpha} \inf_{\delta > \alpha} p(x_\delta)$.

Remark : Every normal bi-l.c. TVS is pseudonormal.

Clearly every γ-complete bi-l.c. TVS is γ-sequentially complete; but the converse is not necessarily true as illustrated in the following

Example 2.3 : Let $X_b \equiv (\ell^1, ||\cdot||_1, \sigma(\ell^1, m_o))$. By Proposition 1.4.3(i), norm convergent and $\sigma(\ell^1, m_o)$-convergent sequences are the same in ℓ^1 and therefore these two topologies have the same bounded sets in ℓ^1 and the space $(\ell^1, \sigma(\ell^1, m_o))$ is sequentially complete. Consequently, X_b is γ-sequentially complete. But $(\ell^1, \sigma(\ell^1, m_o))$ is not quasi-complete since by Proposition 1.4.4(ii), $\beta(m_o, \ell^1)$ is not compatible with the dual pair $\langle m_o, \ell^1 \rangle$. From this it follows that X_b is not γ-complete.

Concerning subsets of a bi-l.c. TVS X_b, we have

Definition 2.4 : A subset B of a bi-l.c. TVS X_b is said to be (i) γ-closed if all γ-limit points of γ-convergent nets contained in B, belong to B, that is, if for some net $\{x_\delta\} \subset B$, $x_\delta \xrightarrow{\gamma} x$ then $x \in B$; (ii) γ-dense (resp. γ-sequentually dense) in X_b provided for each $x \in X$, there is a net $\{x_\delta\}$ (resp. sequence $\{x_n\}$) in B such that $x_\delta \xrightarrow{\gamma} x$ (resp. $x_n \xrightarrow{\gamma} x$); (iii) γ-bounded if for any null sequence $\{\varepsilon_n\}$ of scalars (that is, $\varepsilon_n \to 0$) and $\{x_n\} \subset B$, $\varepsilon_n x_n \xrightarrow{\gamma} 0$; (iv) γ-compact if every net in B has a subnet γ-converging to a point of B; and (v) γ-boundedly compact if every γ-bounded net in B has a subnet γ-converging to a point of B.

Remark : Clearly every T_2-closed set is γ-closed and every γ-compact set is γ-boundedly compact.

Coming to linear mappings defined on bi-l.c. TVS, we introduce

Definition 2.5 : Let $X_b \equiv (X, T_1, T_2)$ and $Y_b \equiv (Y, \tau_1, \tau_2)$ be two bi-locally convex spaces. A linear mapping $f : X_b \to Y_b$ is said to be (i) γ-γ-continuous if $f(x_\delta) \to f(x)$ in Y_b whenever $x_\delta \xrightarrow{\gamma} x$ in X_b, and (ii) a γ-γ-isomorphism if f is one-to-one, onto and has γ-γ-continuous inverse. A linear mapping f from a bi-l.c. TVS X_b to an l.c. TVS (Y, τ) (resp. to the field of scalars $\mathbb{K}$) is called (iii) a γ-continuous linear mapping (resp. γ-continuous linear functional) provided $f(x_\delta) \to f(x)$ in Y (resp. in $\mathbb{K}$) whenever $x_\delta \xrightarrow{\gamma} x$ in X_b.

Definition 2.6 : The family of all γ-continuous linear functionals on X_b which is a vector space with usual pointwise addition and scalar multiplication, is termed as the γ-dual of the bi-l.c. TVS X_b and is denoted by X_γ^*. Similarly, the γ-sequential dual X_γ^+ of X_b is defined as the class of all γ-sequentially continuous linear functionals on X, that is,
$X_\gamma^+ = \{f \in X' : f(x_n) \to f(x) \text{ in } \mathbb{K} \text{ whenever } x_n \xrightarrow{\gamma} x\}$.

Note : Throughout we shall denote by X_i^* and X_i^+ respectively the topological and sequential duals of (X, T_i), $i = 1, 2$.

Lastly, we have

Definition 2.7 : A bi-l.c. TVS X_b is said to be saturated if $X_\gamma^* = X_1^*$.

Remark : The bi-l.c. TVS $(\ell^1, ||\cdot||_1, \sigma(\ell^1, m_o))$ is a saturated bi-l.c. TVS.

x_{β_δ}, $\beta_\delta \geq \delta$ such that

$$p(x_{\beta_\delta}) < a+\varepsilon$$

$$\Longrightarrow \quad x_{\beta_\delta} \in S_{a+\varepsilon,p}, \ \forall \ \delta \in \Lambda .$$

Since $S_{a+\varepsilon,p}$ is T_2-closed and $\{x_{\beta_\delta}\}$ being a subnet of $\{x_\delta\}$, converges to x in T_2, it follows that $x \in S_{a+\varepsilon,p}$. Consequently,

$$p(x) \leq a+\varepsilon .$$

Since ε is arbitrary, we have

$$p(x) \leq \underline{\lim}_{\delta}\, p(x_\delta)$$

This establishes the result.

Before passing to our next result on normal bi-l.c. TVS let us pause a while to characterize the (m) property of a bi-l.c. TVS.

Proposition 3.2 : A bi-l.c. TVS X_b satisfies the property (m) if and only if T_2-closure of each T_1-bounded set is T_1-bounded.

Proof : Let X_b satisfy the property (m) and let B be a T_1-bounded subset of X. Then by hypothesis, there exists a balanced, convex, T_1-bounded and T_2-closed set B_1 of X such that $B \subset B_1$.

Since B_1 is T_2-closed it follows by taking the closure relative to the topology T_2 that T_2-closure of B is T_1-bounded.

For converse, we may take without loss of generality a balanced, convex, T_1-closed and T_1-bounded set B. Since T_2-closure of B is balance, convex, T_1-bounded and T_2-closed, this closure is the desired set containing B and hence X_b satisfies the property (m).

Let us now prove

Proposition 3.3 : A normal bi-l.c. TVS X_b possesses the property (m).

Proof : Let B be a T_1-bounded subset of X. Then for $p \in \mathcal{D}_1$, there exists a number $\alpha > 0$ such that

$$p(x) \leq \alpha, \quad \forall\, x \in B$$

$$\Longrightarrow B \subset S_{\alpha,p}$$

Since $S_{\alpha,p}$ is a T_2-closed set by Proposition 3.1 it follows that T_2-closure $\bar{B}$ of B is also contained in $S_{\alpha,p}$, that is,

$$p(x) \leq \alpha, \ \forall\, x \in \bar{B}.$$

Hence $\bar{B}$ is T_1-bounded and therefore by Proposition 3.2, X_b satisfies the property (m).

Concerning the equivalence of two topologies of a bi-l.c. TVS, we have

Proposition 3.4 : Let $X_b \equiv (X, T_1, T_2)$ be a normal bi-l.c. TVS. Then the following statements are true

(i) If (X, T_2) is barrelled, then the topologies T_1 and T_2 are equivalent.

(ii) If (X, T_2) is an infrabarrelled space and the topologies T_1 and T_2 define the same bounded sets in X, then the topologies T_1 and T_2 are equivalent.

Proof : (i) For proving $T_1 \approx T_2$, it suffices to show $T_1 \subset T_2$ since the other inclusion is true by definition. So, let u

be a T_1-neighbourhood of origin in X. By normal character of X_b, there exists a family $\mathcal{D}_1$ of seminorms generating T_1 such that $S_{\alpha,p} = \{x \in X : p(x) \leq \alpha\}$ is T_2-closed for each $p \in \mathcal{D}_1$ and $\alpha > 0$. There exists v, such that

$$v = \bigcap_{i=1}^{n} \{p_i(x) \leq \alpha_i\} \text{ and}$$

$v \subset u$. Since v is a barrel in (X,T_2) it follows that u contains a T_2-neighbourhood at origin, namely, v. This establishes (i).

(ii) If u is a T_1-neighbourhood of origin then as observed in the first part, u contains a T_2-barrel v which is also a T_1-neighbourhood of origin. Since T_1 and T_2 bounded sets in X are the same, it follows that v is a bornivorous barrel and hence a T_2-neighbourhood at origin. Thus $T_1 \subset T_2$ and the proof is complete.

It is obvious that a normal bi-l.c. TVS X_b is always quasinormal. But the converse implication is true in the following form

<u>Proposition</u> 3.5 : Given a quasinormal bi-l.c. TVS $X_b \equiv (X,T_1,T_2)$ there exists a locally convex topology T_1' on X equivalent to T_1 such that $X_b' \equiv (X,T_1',T_2)$ is a normal bi-l.c. TVS.

<u>Proof</u> : Since X_b is quasinormal there is a family $\mathcal{D}_1$ of seminorms generating T_1 such that for each $p \in \mathcal{D}_1$ we get $q \in \mathcal{D}_1$ satisfying

$$x_\delta \to x \text{ in } (X,T_2) \implies p(x) \leq \underline{\lim}_{\delta}\, q(x_\delta),$$

where $\{x_\delta\}$ is a net in X and $x \in X$. Write

$$v_q = \{x \in X : q(x) \leq 1\}$$

and define

$$Q_p(x) = \sup \{|f(x)| : f \in X_2^* \cap v_q^o\}$$

where v_q^o is the polar of v_q in X_1^*. Clearly, $Q_p(x) < \infty$, for each $x \in X$ and indeed defines a seminorm on X. Let T_1' be the locally convex topology generated by the family $\mathcal{D}_1' = \{Q_p : p \in \mathcal{D}_1\}$ of these seminorms. It is obvious that

$$Q_p(x) \leq q(x), \ \forall\ x \in X.$$

(Indeed, $q(x) = \sup \{|f(x)| : f \in v_q^o\}$) and so $T_1' \subset T_1$.

To establish the reverse inclusion, let,

$$v_p = \{x \in X : p(x) \leq 1\}.$$

Then $\bar{v}_q \subset v_p$, where $\bar{v}_q$ denotes the T_2-closure of v_q in X. Indeed, if $x \in \bar{v}_q$, then there exists a net $\{x_\delta\}$ in v_q such that x_δ converges to x relative to the topology T_2. Consequently,

$$p(x) \leq \underset{\delta}{\underline{\lim}}\ q(x_\delta) \leq 1.$$

and so $x \in V_p$.

Now for $x \in X$ with $p(x) \neq 0$ and arbitrary $\varepsilon > 0$, the point $y = \frac{(1+\varepsilon)x}{p(x)} \notin \bar{v}_q$. Since $\bar{v}_q$ is absolutely convex and T_2-closed by Theorem 1.2.6, there exists an $f \in X_2^*$ such that

$$|f(x)| \leq 1 < |f(y)|, \ \forall\ x \in \bar{v}_q.$$

Consequently,

$$f \in (\bar{v}_q)^o \subset v_q^o$$

and

$$1 < |f(\frac{(1+\varepsilon)x}{p(x)})|$$

or, $$p(x) < (1+\varepsilon)\ Q_p(x)$$

$$\implies p(x) \leq Q_p(x),\ \forall\ x \in X$$

as ε is arbitrary. Thus the topologies T_1 and T_1' are equivalent.

In order to show that the bi-l.c. TVS $X_b' \equiv (X, T_1', T_2)$ is normal, consider a net $\{x_\delta : \delta \in \Lambda\}$ and a point x in X such that $x_\delta \to x$ in (X, T_2). Then for p and q as above and $\varepsilon > 0$, we can find $f \in X_2^* \cap V_q^o$ such that

$$Q_p(x) - \frac{\varepsilon}{2} < |f(x)|$$

$$\leq |f(x-x_\delta)| + |f(x_\delta)|,\ \forall\ \delta \in \Lambda .$$

Now there exists $\delta_o \equiv \delta_o(\varepsilon, f)$ such that

$$|f(x-x_\delta)| \leq \frac{\varepsilon}{2},\ \forall\ \delta \geq \delta_o.$$

Hence,

$$Q_p(x) \leq \varepsilon + Q_p(x_\delta),\quad \forall\ \delta \geq \delta_o$$

$$\implies Q_p(x) \leq \underline{\lim_{\delta}}\ Q_p(x_\delta).$$

Thus (X, T_1', T_2) is a normal bi-l.c. TVS and this completely establishes the result.

Remark : It is clear from the Proposition 3.5 that a quasinormal bi-l.c. TVS possesses all the properties of a normal bi-l.c. TVS which remain invariant under the equivalence of topologies.

We conclude this section with the following property of a saturated bi-l.c. TVS.

<u>Proposition</u> 3.6 : A saturated bi-l.c. TVS $X_b \equiv (X, T_1, T_2)$ is a pseudonormal bi-l.c. TVS.

<u>Proof</u> : Let $\{x_\delta : \delta \in \Lambda\}$ be a net in X_b, γ-converging to a point x_o of X and $p \in \mathcal{D}_{T_1}$. Then by Proposition 1.2.8, there exists an $f \in X_1^*$ such that

$$|f(x)| \leq p(x), \ \forall\, x \in X$$

and

$$f(x_o) = p(x_o).$$

As $X_1^* = X_\gamma^*$, the net $\{x_\delta\}$ converges to x_o in $\sigma(X, X_1^*)$. Hence for given $\varepsilon > 0$ there exists $\delta_o \in \Lambda$ such that

$$|f(x_\delta - x_o)| < \varepsilon, \ \forall\, \delta > \delta_o.$$

$$\begin{aligned} \Longrightarrow |f(x_o)| &\leq |f(x_\delta - x_o)| + |f(x_\delta)| \\ &< \varepsilon + |f(x_\delta)|, \quad \forall\, \delta \geq \delta_o \\ &< \varepsilon + p(x_\delta), \quad \forall\, \delta \geq \delta_o \end{aligned}$$

$$\Longrightarrow p(x_o) \leq \underline{\lim_{\delta}}\, p(x_\delta).$$

Hence the space X_b is pseudonormal.

4. γ-<u>Boundedness</u>, γ-<u>Compactness</u> <u>and</u> γ-γ-<u>Isomorphisms</u> :

In this section we have characterized γ-bounded and γ-compact sets in terms of their corresponding properties relative to the given topologies and have shown that γ-boundedness and γ-compactness remain invariant under γ-γ-continuity, whereas γ-boundedly compactness is not preserved for sets. We also derive conditions for the γ-γ continuity of the inverse of a γ-γ continuous linear map.

We now begin with

Proposition 4.1 : A set B in a bi-l.c. TVS $X_b \equiv (X, T_1, T_2)$ is γ-bounded if and only if it is T_1-bounded.

Proof : Let B be a γ-bounded set in X_b. Choose a sequence $\{x_n\}$ in B and a sequence $\{\varepsilon_n\}$ of scalars with $\varepsilon_n \to 0$. Then $\sqrt{|\varepsilon_n|} \to 0$ and as B is γ-bounded,

$$\sqrt{|\varepsilon_n|}\, x_n \overset{\gamma}{\to} 0$$

$$\Longrightarrow \{\sqrt{|\varepsilon_n|}\, x_n\} \text{ is } T_1\text{-bounded and hence}$$

$$\sqrt{|\varepsilon_n|}\,\sqrt{|\varepsilon_n|}\, x_n \to 0 \text{ relative to } T_1$$

$$\Longrightarrow \varepsilon_n x_n \to 0 \text{ relative to } T_1$$

By Proposition 1.2.4(i), B is T_1-bounded.

Converse is trivially true because T_1-sequential convergence always implies γ-sequential convergence.

Concerning γ-compactness of a set, we have

Proposition 4.2 : A set B in a bi-l.c. TVS $X_b \equiv (X, T_1, T_2)$ is γ-compact if and only if it is T_1-bounded and T_2-compact.

Proof : For proving the result we need prove the necessity as the sufficiency part clearly follows by definition. So, let us consider a γ-compact set B in X_b. Obviously, B is T_2-compact. Also, we claim that B is T_1-bounded. For, if B is not T_1-bounded we can find a $p \in \mathcal{D}_{T_1}$ and a sequence $\{x_n\}$ of distinct elements in B such that

$$p(x_n) > 2^n.$$

We now show that no subnet of $\{x_n\}$ having infinitely distinct terms γ-converges to a point of B. Let us therefore consider a subnet $\{x_\alpha\}$ of the sequence $\{x_n\}$, γ-converging to a point of B. Since $\{x_\alpha\}$ is T_1-bounded, there exists a scalar $\lambda > 0$ such that

$$p(x_\alpha) < \lambda, \forall \alpha.$$

Choose n_o as the last positive integer for which $\lambda < 2^{n_o}$. Then

$$p(x_\alpha) < 2^{n_o}, \forall \alpha.$$

Hence the net $\{x_\alpha\}$ can assume only finite values $x_1, \ldots, x_{n_o-1}$. This contradicts that B is γ-compact and the result is established.

Remark: It is clear from Proposition 4.2 that γ-compactness coincides with γ-boundedly compactness on T_1-bounded sets. However, a γ-boundedly compact set may be T_1-unbounded; for instance

Example 4.3: Consider the bi-l.c. TVS $X_b \equiv (\ell^1, ||\cdot||_1, \sigma(\ell^1, \phi))$. The set $A = \{ne^n : n \geq 1\} \cup \{0\}$ is γ-boundedly compact because every $||\cdot||_1$-bounded net in A has a subnet converging to 0. But A is $||\cdot||_1$-unbounded.

Next, we have

Proposition 4.4: A γ-γ-continuous linear mapping f from a bi-l.c.TVS $X_b \equiv (X, T_1, T_2)$ into another bi-l.c. TVS $Y_b \equiv (Y, \tau_1, \tau_2)$ maps a γ-bounded set into a γ-bounded set.

Proof: Let A be a γ-bounded subset X_b. To show that f(A)

is γ-bounded in Y_b, consider a sequence $\{y_n\}$ in $f(A)$ and $\{\varepsilon_n\}$ in $\mathbb{K}$ such that $\varepsilon_n \to 0$. Then $y_n = f(x_n)$, $x_n \in A$, $n \geq 1$ and so by γ-boundedness of A,

$$\varepsilon_n x_n \overset{\gamma}{\to} 0$$

consequently, by γ-γ continuity of f

$$\varepsilon_n\ f(x_n) \overset{\gamma}{\to} 0.$$

Thus $f(A)$ is γ-bounded in Y_b and the result is established.

Proposition 4.5: A γ-γ-continuous linear mapping f from a bi-l.c. TVS $X_b \equiv (X, T_1, T_2)$ into another bi-l.c. TVS $Y_b \equiv (Y, \tau_1, \tau_2)$ maps a γ-compact subset A into a γ-compact subset of $f(A)$.

Proof: Applying similar arguments as in the preceding proposition, the result follows immediately from the definitions of γ-compactness and γ-γ-continuity of the linear mapping f.

Note: It is worthmentioning that the γ-boundedly compactness is not preserved under the γ-γ-continuous linear map, for instance, we have

Example 4.6: Let F be a linear mapping from $X_b \equiv (\ell^1, ||\cdot||_1, \sigma(\ell^1, \phi))$ to $Y_b \equiv (\ell^1, ||\cdot||_1, \sigma(\ell^1, m_o))$ defined by $F(\{x_n\}) = \{\frac{x_n}{n}\}$. F is clearly a γ-γ-continuous linear and the set $A = \{ne^n\} \cup \{0\}$ is γ-boundedly compact in X_b as seen in Example 4.3. Since γ-boundedly compactness coincides with γ-compactness on T_1-bounded sets by a remark after Proposition 4.2, $F(A) = \{e^n\} \cup \{0\}$ is not γ-boundedly compact in Y_b as it is not $\sigma(\ell^1, m_o)$-compact.

For the last result of this section, we need

Proposition 4.7: Let $X_b \equiv (X, T_1, T_2)$ and $Y_b \equiv (Y, \tau_1, \tau_2)$ be two bi-l.c. TVS such that (X, T_1) is a bornological space. Then a γ-γ-continuous linear mapping $F: X_b \to Y_b$ is also a continuous linear mapping from (X, T_1) to (Y, τ_1).

Proof: In view of Proposition 1.2.23 it suffices to show that F maps a T_1-bounded subset of X into a τ_1-bounded subset of Y. Equivalently, γ-bounded subset of X_b is mapped into a γ-bounded subset of Y_b by virtue of Proposition 4.1. However, the last statement follows from Proposition 4.4. Hence F is T_1-τ_1 continuous and the result is established.

We have the final result of this section in the form of

Theorem 4.8: Let $X_b \equiv (X, T_1, T_2)$ and $Y_b \equiv (Y, \tau_1, \tau_2)$ be two bi-locally convex spaces such that (X, T_1) is bornological and fully complete and (Y, τ_1) is a barrelled space. Further, assume that X_b is γ-boundedly compact. Then a γ-γ continuous linear bijection $F: X_b \to Y_b$ is a topological isomorphism from (X, T_1) to (Y, τ_1) and is also a γ-γ isomorphism. Further, Y_b is also γ-boundedly compact.

Proof: By Proposition 4.7, F is a T_1-τ_1 continuous linear mapping and hence by Theorem 1.2.25 F is a topological isomorphism from (X, T_1) to (Y, τ_1).

For showing γ-γ continuity of F^{-1}, let us consider a net $\{y_\delta\}$ in Y such that $y_\delta \xrightarrow{\gamma} 0$. Write

$$y_\delta = F(x_\delta),\ x_\delta \in X.$$

It suffices to show $x_\delta \to 0$ relative to the topology T_2, as $\{x_\delta\}$ being the continuous image of a τ_1-bounded net $\{y_\delta\}$ under F^{-1} is T_1-bounded. So, let us assume the contrary, i.e., $x_\delta \not\to 0$ in (X,T_2). Hence there exists a subnet $\{x_{\delta'}\}$ of $\{x_\delta\}$ such that no subnet of $\{x_{\delta'}\}$ converges to zero in (X,T_2) (cf. [66], p. 74). Since $\{x_{\delta'}\}$ is γ-bounded and X_b is γ-boundedly compact there exists a subnet $\{x_{\delta''}\}$ of $\{x_{\delta'}\}$ and $x \in X$ such that

$$x_{\delta''} \xrightarrow{\gamma} x$$

$$\Longrightarrow \quad F(x_{\delta''}) \to F(x) \text{ relative to the topology } \tau_2.$$

Since $\{F(x_{\delta''})\}$ is a subnet of $\{y_\delta\}$ which converges to zero relative to T_2, it follows that $F(x) = 0$. Consequently $x=0$ as F is an injection. Hence

$$x''_\delta \to 0 \text{ in } (X,T_2),$$

which is a contradiction. Thus F is a γ-γ isomorphism.

Lastly to establish γ-boundedly compactness of Y_b, let us consider a γ-bounded net $\{y_\delta\}$ in Y_b. If $y_\delta = F(x_\delta)$, $x_\delta \in X$, then by γ-γ continuity of F^{-1}, $\{x_\delta\}$ is γ-bounded. Since X_b is γ-boundedly compact, there exists a subnet $\{x_{\delta'}\}$ of $\{x_\delta\}$ which γ-converges to a point x in X. Consequently, $\{F(x_{\delta'})\}$ γ-converges to $F(x)$ in Y_b and γ-boundedly compactness of Y_b follows.

<u>Note</u> (i) Though γ-γ continuous image of a γ-boundedly compact subset is not γ-boundedly compact as seen in Example 4.6, the

preceding theorem assures that the γ-γ isomorphic image of a γ-boundedly compact space is γ-boundedly compact.

(ii) Further, we like to mention here that γ-γ isomorphism from a bi-l.c. TVS X_b to another bi-l.c. TVS Y_b is not necessarily isomorphisms from (X,T_1) to (Y,τ_1) and from (X,T_2) to (Y,τ_2) separately. For examples, the identity mappings from $(\ell^1, \tau(\ell^1, c_o), \sigma(\ell^1,\phi))$ to $(\ell^1, ||\cdot||_1, \sigma(\ell^1, c_o))$ and from $(\ell^\infty, \tau(\ell^\infty, \ell^1), \sigma(\ell^\infty,\phi))$ to $(\ell^\infty, ||\cdot||_\infty, \sigma(\ell^\infty, \ell^1))$ are γ-γ topological isomorphisms; but in each case they are not separately topological isomorphisms. Indeed this follows from the fact that the spaces $(\ell^\infty, \tau(\ell^\infty, \ell^1))$ and $(\ell^1, \tau(\ell^1, c_o))$ are not infrabarrelled (cf. Propositions 1.4.4 (i) and Proposition 1.4.3(ii)).

5. <u>Duals of Bi-locally Convex Spaces</u>:

In this section, we mainly discuss the relations among the three duals, namely, X_1^*, X_2^* and X_γ^* of a bi-l.c. TVS $X_b \equiv (X,T_1,T_2)$, as defined in Definition 2.6. We start from a simple result, namely,

<u>Proposition</u> 5.1: For a bi-l.c. TVS $X_b \equiv (X,T_1,T_2)$

$$X_2^* \subset X_\gamma^* \subset X_\gamma^+ \subset X_1^+$$

In addition, if (X,T_1) is also Mazur, $X_2^* \subset X_\gamma^* \subset X_1^*$.

<u>Proof</u>: The inclusion $X_2^* \subset X_\gamma^*$ follows immediately from the definition of γ-convergence; whereas $X_\gamma^* \subset X_\gamma^+$, is trivially true.

For showing the inclusion $X_\gamma^+ \subset X_1^+$, let $f \in X_\gamma^+$ and $\{x_n\}$ be a sequence in X such that $x_n \to x$ in (X,T_1). As $T_2 \subset T_1$,

$x_n \xrightarrow{\gamma} x$ and therefore $f(x_n) \to f(x)$, which in turn implies $f \in X_1^+$. Thus $X_\gamma^+ \subseteq X_1^+$.

If (X,T_1) is a Mazur space, then $X_1^* = X_1^+$ and hence the last part follows.

An interesting property possessed by X_γ^* and X_γ^+ is contained in

Proposition 5.2: For a bi-l.c.TVS $X_b \equiv (X,T_1,T_2)$, X_γ^* and X_γ^+ are closed subspaces of $(X_1^+, \beta(X_1^+,X))$. If (X,T_1) is Mazur, X_γ^* and X_γ^+ are closed subspaces of $(X_1^*, \beta(X_1^*,X))$.

Proof: Let us first observe that the T_1-bounded and $\sigma(X,X_1^+)$-bounded sets in X are the same. Indeed, if A is T_1-bounded, then for any sequence $\{x_n\}$ in A and $\varepsilon_n \to 0$ in $\mathbb{K}$, $\varepsilon_n x_n \to 0$ relative to T_1 and so $\varepsilon_n f(x_n) \to 0$, for each $f \in X_1^+$. Consequently, A is $\sigma(X,X_1^+)$-bounded. Conversely, if A is $\sigma(X,X_1^+)$-bounded, then it is $\sigma(X,X_1^*)$-bounded and so T_1-bounded by Proposition 1.2.11.

For proving the closed character of X_γ^* in $(X_1^+,\beta(X_1^+,X))$ let us consider an f in the $\beta(X_1^+,X)$-closure $\bar{X}_\gamma^*$ of X_γ^* in X_1^+; and a net $\{x_\delta : \delta \in \Lambda\}$ in X, which γ-converges to a point x of X. For showing $f(x_\delta) \to f(x)$, consider the $\sigma(X,X_1^+)$-bounded set B defined by

$$B = \{x_\delta\} \cup \{x\}.$$

Then for a given $\varepsilon > 0$, there exists $f_o \in X_\gamma^*$ such that

$$p_B(f-f_o) < \frac{\varepsilon}{3}.$$

Also there exists δ_o depending on ε and f such that

$$|f_o(x_\delta) - f_o(x)| < \frac{\varepsilon}{3}, \ \forall \ \delta \geq \delta_o.$$

Hence,

$$\begin{aligned} |f(x_\delta)-f(x)| &\leq |f(x_\delta)-f_o(x_\delta)|+|f_o(x_\delta)-f_o(x)| + |f_o(x)-f(x)| \\ &\leq p_B(f-f_o) + |f_o(x_\delta) - f_o(x)| + p_B(f-f_o) \\ &< \varepsilon \qquad , \forall \ \delta \geq \delta_o . \end{aligned}$$

Thus, $f \in X^*_\gamma$ and therefore $\bar{X}^*_\gamma \subset X^*_\gamma$. Consequently, X^*_γ is a $\beta(X_1^+,X)$-closed subspace of X_1^+.

Replacing net convergence by sequential convergence, we can similarly establish that X^+_γ is a $\beta(X_1^+,X)$ closed subspace of X_1^+. The last part is immediate from the definition of a Mazur space.

The above proposition yields

<u>Proposition</u> 5.3: Let $X_b \equiv (X,T_1,T_2)$ be a bi-l.c. TVS with (X,T_1) as a Mazur space. Then X^*_γ is a complete subspace of $(X_1^*,\beta(X_1^*,X))$.

<u>Proof</u>: Since the strong dual $(X_1^*,\beta(X_1^*,X))$ of (X,T_1) is complete by Proposition 1.2.21, the space X^*_γ being a closed subspace of $(X_1^*,\beta(X_1^*,X))$ is therefore complete.

<u>Proposition</u> 5.4: If (X,T_1) is a semireflexive Mazur space then for any locally convex topology T_2 inferior to T_1, the bi-l.c. TVS $X_b \equiv (X,T_1,T_2)$ is saturated.

<u>Proof</u>: For proving the result, it suffices to show that

$X_1^* \subset X_\gamma^*$. So, let us assume the contrary. Then there exists an $f_o \in X_1^*$ such that $f_o \notin X_\gamma^*$. Since X_γ^* is a $\beta(X_1^*,X)$-closed subspace of X_1^*, by Proposition 1.2.7 there exists a non-zero $F \in X_{11}^{**} = (X_1^*, \beta(X_1^*,X))^*$ such that

$$F(f) = 0 \quad , \quad \forall \quad f \in X_\gamma^*$$

and

$$F(f_o) = 1.$$

By the semireflexivity of (X,T_1), there is a nonzero x in X such that

$$\begin{aligned} & F(f) = f(x) \quad , \quad \forall \quad f \in X_\gamma^* \\ \Longrightarrow \quad & f(x) = 0 \quad , \quad \forall \quad f \in X_\gamma^* \\ \Longrightarrow \quad & x = 0 \end{aligned}$$

since X_γ^* is total over X. Thus we arrive at a contradiction and so $X_1^* = X_\gamma^*$, that is, X_b is saturated.

Restricting the bi-l.c. TVS X_b, we obtain

<u>Proposition</u> 5.5: Let $X_b \equiv (X,T_1,T_2)$ be a bi-l.c. TVS such that (X,T_1) is a Mazur space and X_b satisfies the property (m). Then X_2^* is dense in X_γ^* relative to the strong topology $\beta(X_1^*,X)$.

<u>Proof</u>: By Proposition 5.1 and 5.2, $\bar{X}_2^* \subset X_\gamma^*$ and so we need show the other inclusion. Therefore, consider an $f \in X_\gamma^*$ and choose $\delta > 0$. Then for a given T_1-bounded set B which we may also choose to be balanced, convex and T_2-closed in view of our hypothesis, there exists a balanced, convex and T_2-closed

neighbourhood u of 0 in (X,T_2) such that

$$|f(x)| < \delta\ ,\quad \forall\ x \in B \cap u\ .$$

For, if this is not true, then there exists $\varepsilon > 0$ such that for each $u \in \mathcal{U}_{T_2}(X)$ there is a $x_u \in B \cap u$ with $|f(x_u)| \geq \varepsilon$. Now consider the net $\{x_u : u \in \mathcal{U}_{T_2}(X)\}$ where the neighbourhood system $\mathcal{U}_{T_2}(X)$ is directed by the usual relation

$$u \geq v \text{ if and only if } u \subseteq v.$$

Clearly $x_u \xrightarrow{\gamma} 0$ but $f(x_u) \not\to 0$. This implies f is not a member of X^*_γ, which is not true.

Consequently,

$$f \in \delta(B \cap u)^{\circ},$$

where the polar is taken relative to the dual pair $\langle X, X_1^* \rangle$.
Also applying theorem 1.2.12 and an elementary polar operation for union and intersection

$$(B \cap u)^{\circ} = (B^{\circ\circ} \cap u^{\circ\circ})^{\circ}$$

(*) $$= ((B^{\circ} \cup u^{\circ})^{\circ})^{\circ} \quad \text{(cf. [57] p. 191)}$$

Since B° is $\sigma(X_1^*,X)$-closed and u° is $\sigma(X_1^*,X)$-compact, $B^{\circ}+u^{\circ}$ is $\sigma(X_1^*,X)$-closed (cf. [57] p 145). Also it is balanced and convex and so by Bi-polar theorem, we get

$$(B^{\circ}+u^{\circ})^{\circ\circ} = B^{\circ}+u^{\circ}.$$

Now,

$$B^{\circ} \cup u^{\circ} \subset B^{\circ}+u^{\circ}$$

(**) $$\Longrightarrow (B^{\circ} \cup u^{\circ})^{\circ\circ} \subseteq (B^{\circ}+u^{\circ})^{\circ\circ} = B^{\circ}+ u^{\circ}.$$

Therefore, from (*) and (**)

$$(B \cap u)^{\circ} \supseteq B^{\circ} + u^{\circ}$$

Hence,

$$f \in \delta(B^{\circ} + u^{\circ})$$

Hence, there exists $g \in \delta u^{\circ}$ such that

$$f - g \in \delta B^{\circ}$$

$$\text{or,} \quad p_B(f-g) \leq \delta \ .$$

But $g \in \delta u^{\circ}$ implies that $g \in X_2^*$. Hence X_2^* is dense in X_γ^* relative to the topology $\beta(X_1^*, X)$. This completes the proof.

Remark: Since a normal bi-l.c. TVS X_b satisfies the property(m) (cf. Proposition 3.3), the result holds good for such bi-l.c. TVS $X_b \equiv (X, T_1, T_2)$ with (X, T_1) as a Mazur space. Also, from Proposition 3.5, we get

Proposition 5.6: If $X_b \equiv (X, T_1, T_2)$ is a quasinormal space for which (X, T_1) is a Mazur space, then X_2^* is dense in X_γ^* relative to $\beta(X_1^*, X)$.

Proof: By Proposition 3.5, there exists a locally convex topology T_1' equivalent to T_1 such that $X_b' \equiv (X, T_1', T_2)$ is normal and by hypothesis (X, T_1') is Mazur. Since X_b and X_b' have the same γ-duals and (X, T_1) and (X, T_1') have the same topological duals, by the preceding remark X_2^* is $\beta(X_1^*, X)$-dense in X_γ^*.

Note: The importance of Proposition 5.5 lies in finding the γ-duals of bi-locally convex spaces; for instance, let us

consider the bi-l.c. TVS $X_b \equiv (\ell^1, ||\cdot||_1, \sigma(\ell^1, c_o))$, (cf. Remark after Definition 2.1). It is known that $||\cdot||_1$ and $\sigma(\ell^1, c_o)$ have the same bounded sets, and so X_b satisfies the property (m) by virtue of Proposition 3.2. Therefore, applying Proposition 5.5, we get

$$X_\gamma^* = \bar{X}_2^* = \bar{c}_o$$

where the closure of c_o is taken relative to $\beta(\ell^\infty, \ell^1) \equiv ||\cdot||_\infty$. But c_o is $||\cdot||_\infty$-closed and hence $X_\gamma^* = c_o$ in this case. Likewise, one can easily find the γ-duals of $(\ell^1, \sigma(\ell^1, \ell^\infty), \sigma(\ell^1, m_o))$ and $(c_o, \eta(c_o, \ell^1), \sigma(c_o, \phi))$ as ℓ^∞ and ℓ^1 respectively.

A generalization of a result of Alexiewicz and Semadeni (cf. [14] , Proposition 1.5, p. 124), useful in the sequel, is contained in

<u>Proposition</u> 5.7: Let $X_b \equiv (X, T_1, T_2)$ be a normal bi-l.c. TVS and $v = \{x \in X : p(x) \leq 1\}$ for $p \in \mathcal{P}_{T_1}$. If v^o is the polar of v in X_γ^*, then

$$p(x) = \sup \{|f(x)| : f \in v^o\}.$$

<u>Proof</u>: Since for $x \in X$, with $p(x) \neq 0$, $\frac{x}{p(x)} \in v$, the inequality

$$(*) \qquad \sup \{|f(x)| : f \in v^o\} \leq p(x)$$

clearly follows. For the reverse inequality, let us observe that v is balanced, convex and T_2-closed (cf. Proposition 3.1) and for $x \in X$ with $p(x) \neq 0$ and $\varepsilon > 0$, $\frac{(1+\varepsilon)x}{p(x)} \notin v$. Applying Theorem 1.2.6 as in Proposition 3.5, we get

$$(**) \qquad p(x) \leq \sup \{|f(x)| : f \in V^{\circ}\}$$

Hence from (*) and (**) equality follows.

Note: Let us point out that if V° denotes the polar of V in X_1^* for a normal bi-l.c. TVS, then looking at the proof of the above proposition, one can also derive the following equalities

$$\begin{aligned} p(x) &= \sup \{|f(x)| : f \in V^{\circ} \cap X_2^*\} \\ &= \sup \{|f(x)| : f \in V^{\circ} \cap X_\gamma^*\} \end{aligned}$$

However, in case of quasinormal bi-l.c. TVS the following inequalities follow from the proof of the Proposition 3.5 - for each $p \in \mathcal{D}_1$ there is a $q \in \mathcal{D}_1$ such that

$$\begin{aligned} p(x) &\leq \sup \{|f(x)|: f \in V_q^{\circ} \cap X_2^*\} \\ &\leq \sup \{|f(x)|: f \in V_q^{\circ} \cap X_\gamma^*\} \end{aligned}$$

The final result of this chapter which provides sufficient conditions for the induced topology $\beta(X_1^*,X)|_{X_\gamma^*}$ to be the same as $\beta(X_\gamma^*,X)$ and is useful for our study on γ-reflexivity in the next chapter, runs as follows

Proposition 5.8: Let $X_b \equiv (X,T_1,T_2)$ be a normal bi-l.c. TVS such that (X,T_1) is a Mazur space and the dual system $\langle X,X_\gamma^*\rangle$ is an M-system. Then

$$\beta(X_\gamma^*,X) = \beta(X_1^*,X)|_{X_\gamma^*},$$

that is, $\beta(X_\gamma^*,X)$ coincides with the topology induced on X_γ^* by $\beta(X_1^*,X)$.

Proof: For proving this result, it is sufficient to show that $\sigma(X,X_\gamma^*)$-and $\sigma(X,X_1^*)$-bounded sets are the same. This would follow if we show that a $\sigma(X,X_\gamma^*)$-bounded set B in X is $\sigma(X,X_1^*)$-bounded, for the other implication is obviously true. Let us therefore consider a $\sigma(X,X_\gamma^*)$-bounded set B and assume that it is not $\sigma(X,X_1^*)$-bounded. Therefore, we can find a $\sigma(X,X_1^*)$-unbounded sequence $\{x_n\}$ in B. Now, define a sequence $\{F_n\}$ of linear functionals on X_γ^* as follows.

$$F_n(f) = f(x_n), \quad \forall \quad f \in X_\gamma^*.$$

We claim that $\{F_n : n \geq 1\}$ is $\beta(X_\gamma^{**}, X_\gamma^*)$- bounded, where X_γ^{**} is the topological dual of $(X_\gamma^*, \beta(X_\gamma^*, X))$. For, if A is a $\beta(X_\gamma^*, X)$-bounded subset of X_γ^* then there exists a $\mu > 0$ such that

$$A \subset \mu\{x_n : n \geq 1\}^{\circ}$$

$$\Longrightarrow \quad |f(x_n)| \leq \mu, \quad \forall \ f \in A \quad \text{and} \quad n \geq 1,$$

$$\Longrightarrow \quad |F_n(f)| \leq \mu, \quad \forall \ f \in A \quad \text{and} \quad n \geq 1,$$

$$\Longrightarrow \quad F_n \in \mu \ A^{\circ}, \ \text{for all } n \geq 1.$$

Now, for a seminorm $p \in \mathcal{D}_{T_1}$, we know that the set $v = \{x \in X : p(x) \leq 1\}$ is absorbing. Hence the polar v° of v in X_γ^* is $\sigma(X_\gamma^*, X)$-bounded and so $\beta(X_\gamma^*, X)$-bounded as the dual system $\langle X, X_\gamma^* \rangle$ is an M-system. Consequently, there exists a scalar $K > 0$ such that

$$\{F_n : n \geq 1\} \subset K v^{\circ}$$

or, equivalently

$$(*) \qquad \sup_{n \geq 1} \ \sup_{f \in v^{\circ}} |F_n(f)| \leq K.$$

Since $p(x_n) = \sup_{f \in V^o} |F_n(f)|$ by Proposition 5.7, it follows that

$$\sup_{n \geq 1} p(x_n) \leq K,$$

that is, $\{x_n\}$ is a bounded sequence in (X, T_1). This contradicts that $\{x_n\}$ is $\sigma(X, X_1^*)$-unbounded and hence our assumption is wrong and therefore, the result follows.

Remark: This result also holds if $X_b \equiv (X, T_1, T_2)$ is a quasi-normal bi-l.c. TVS. Indeed, for each $p \in \mathcal{D}_1$ there exists a $q \in \mathcal{D}_1$ such that

$$p(x) \leq \sup \{ |f(x)| : f \in X_\gamma^* \cap V_q^o \}.$$

So, if we consider V_q in place of V in the proof of the preceding result, we get from (*)

$$\sup_{n \geq 1} \sup_{f \in V_q^o \cap X_\gamma^*} |F_n(f)| \leq K$$

$$\implies \sup_{n \geq 1} p(x_n) \leq K.$$

CHAPTER - 4

γ-REFLEXIVITY

CONTENTS

1. Introduction

This chapter deals with the duality aspect of a bi-locally convex space. Indeed, we introduce here in a natural way the notions of γ-conjugate, γ-semireflexive and γ-reflexive bi-l.c. TVS. After studying some properties of γ-conjugate spaces in the second section, we characterize γ-semireflexive spaces in the third section. The fourth section incorporates results on γ-reflexivity.

2. γ-Conjugate Spaces

Corresponding to a bi-l.c. TVS $X_b \equiv (X, T_1, T_2)$ let us note

(i) $X_2^* \subset X_\gamma^*$; and

(ii) $\sigma(X, X_1^*)$-bounded subsets are also $\sigma(X, X_2^*)$-bounded.

Thus from (ii), we have

$$\beta(X_1^*, X)\,|_{X_2^*} \subset \beta(X_2^*, X).$$

These observations yield in a natural fashion the following

<u>Definition</u> 2.1: The triplet $(X_2^*, \beta(X_2^*, X), \beta(X_1^*, X)\,|_{X_2^*})$ is a bi-l.c. TVS which is known as the <u>first</u> γ-<u>conjugate</u> <u>bi-l.c. TVS</u> or the <u>first</u> γ-<u>conjugate space</u> of X_b and is <u>denoted by</u> γ-X_b. Likewise, the <u>second</u> γ-<u>conjugate</u> space of X_b is defined as the first γ-conjugate space of γ-X_b and is <u>denoted</u> by γ^2-X_b. In other words,

$$X_b \equiv (X, T_1, T_2),$$

$$\gamma\text{-}X_b \equiv (X_2^*, \beta(X_2^*, X), \beta(X_1^*, X)|_{X_2^*}), \text{ and}$$

$$\gamma^2\text{-}X_b \equiv (X_{21}^{**}, \beta(X_{21}^{**}, X_2^*), \beta(X_{22}^{**}, X_2^*)|_{X_{21}^{**}})$$

where

$X_{21}^{**} = (X_2^*, \beta(X_1^*, X)|_{X_2^*})^*$ = topological dual of X_2^* relative to the induced topology $\beta(X_1^*, X)|_{X_2^*}$; and

$X_{22}^{**} = (X_2^*, \beta(X_2^*, X))^*$ = topological dual of X_2^* relative to $\beta(X_2^*, X)$

Coming to the results of this section which deal with the structural properties of the γ-conjugate spaces, we have

Proposition 2.2: The γ-conjugate spaces $\gamma\text{-}X_b$ and $\gamma^2\text{-}X_b$ of a bi-l.c. TVS $X_b \equiv (X, T_1, T_2)$ are normal.

Proof: Let us note that the proof would be completely established if we prove the normality of $\gamma\text{-}X_b = (X_2^*, \beta(X_2^*, X), \beta(X_1^*, X)|_{X_2^*})$. Therefore, consider a net $\{f_\delta\}_{\delta \in \Lambda}$ in X_2^* and an $f \in X_2^{**2}$ such that $f_\delta \to f$ relative to $\beta(X_1^*, X)|_{X_2^*}$. Then $f_\delta(x) \to f(x)$, for each x in X. Choose an $\varepsilon > 0$ and a T_2-bounded set B in X. Since

$$p_B(f) = \sup_{x \in B} |f(x)|,$$

there is a $x_o \in B$ such that

$$p_B(f) < |f(x_o)| + \frac{\varepsilon}{2} .$$

Since $f_\delta(x_o) \to f(x_o)$, there is a $\delta_o \in \Lambda$ such that

$$|f_\delta(x_o) - f(x_o)| < \frac{\varepsilon}{2}, \forall \delta > \delta_o$$

Consequently,

$$\begin{aligned} p_B(f) &< |f(x_o)| + \frac{\varepsilon}{2} \\ &\leq |f_\delta(x_o)-f(x_o)| + |f_\delta(x_o)| + \frac{\varepsilon}{2} \\ &< \varepsilon + p_B(f_\delta), \quad \forall \delta > \delta_o \end{aligned}$$

Hence

$$p_B(f) \leq \varliminf_{\delta} p_B(f_\delta)$$

and so the space γ-X_b is normal.

Next, we prove

<u>Proposition</u> 2.3: Let $X_b \equiv (X,T_1,T_2)$ be a bi-l.c. TVS such that (X,T_2) is barrelled. Then the γ-conjugate space γ-X_b is a γ-complete bi-l.c. TVS.

<u>Proof</u>: Let $\{f_\delta\}_{\delta\in\Lambda}$ be a γ-Cauchy net in γ-X_b. Then for a given T_1-bounded subset B of X and an arbitrary $t > 0$, there exists a $\delta_o \in \Lambda$ such that

(*) $$p_B(f_\delta - f_\eta) < t \quad , \forall \delta, \eta \geq \delta_o$$

Also, for each T_2-bounded subset A of X, there exists $\mu_A > 0$ such that

(**) $$\sup_{x\in A} |f_\delta(x)| < \mu_A.$$

From (*) and (**) it follows that $\{f_\delta(x)\}$ is a Cauchy net in $\mathbb{K}$ for each x in X and it is pointwise bounded. Applying Barrel theorem to the space (X,T_2), we can find f in X_2^* such that

(+) $$\lim_{\delta} f_\delta(x) = f(x), \quad \forall x \in X.$$

Clearly, $\{f_\delta : \delta \in \Lambda\} \cup \{f\}$ is $\beta(X_2^*, X)$-bounded by (**). Thus to establish the result completely, we need only show that

(***) $$f_\delta \to f \text{ in } \beta(X_1^*, X)|_{X_2^*} .$$

Now from (*)

$$f_\delta - f_\eta \in tB^o, \quad \forall \ \delta, \eta \geq \delta_o$$

Since tB^o is $\sigma(X_1^*, X)$-closed, we have by (+)

$$f_\delta - f \in t\, B^o, \quad \forall \delta \geq \delta_o .$$

As B is an arbitrary T_1-bounded set in X, (***) follows and hence γ-X_b is γ-complete.

3. γ-<u>Semireflexive Bi-Locally Convex Spaces</u>:

As observed in the last section, we have the conjugate spaces γ-X_b and γ^2-X_b corresponding to a bi-l.c. TVS X_b. Therefore it is natural to seek for notions like semi-reflexivity and reflexivity which relate the space X_b with its second conjugate space. In this section as well as in the following section, we study these notions and term them γ-semireflexivity and γ-reflexivity.

We know that for a subclass of bi-l.c. TVS $X_b \equiv (X, T_1, T_2)$, where (X, T_1) is a Mazur space and X_b satisfies the property (m),

$$\bar{X}_2^* = X_\gamma^*$$

by Proposition 3.5.5, the closure being considered relative to the topology $\beta(X_1^*, X)$. Therefore X_2^* and X_γ^* have the same topological duals relative to the topology $\beta(X_1^*, X)$. In other-words

$$X_{21}^{**} = (X_2^*,\ \beta(X_1^*,X)|_{X_2^*})^*$$
$$= (X_\gamma^*,\ \beta(X_1^*,X)|_{X_\gamma^*})^* = X_{\gamma 1}^{**}$$

Thus for this restricted class of bi-l.c. TVS there is a well-defined linear and one-to-one mapping $J:X \to X_{21}^{**} = X_{\gamma 1}^{**}$, defined by

$$(Jx)f = f(x), \text{ for } x \in X \text{ and } f \in X_\gamma^*.$$

We call this mapping J as the canonical embedding from X_b into X_{21}^{**} and this leads to

<u>Definition</u> 3.1: A bi-l.c. TVS $X_b \equiv (X,T_1,T_2)$ satisfying the property (m) and for which (X,T_1) is a Mazur space, is said to be γ-<u>semireflexive</u> if the canonical mapping $J:X \to X_{21}^{**}$ is onto.

For an example of a γ-semireflexive space, we have

<u>Example</u> 3.2: Consider the bi-l.c. TVS $X_b \equiv (\ell^1,\sigma(\ell^1,\ell^\infty),\sigma(\ell^1,\phi))$. Let us first note that the space $(\ell^1,\sigma(\ell^1,\ell^\infty))$ is a Mazur space and X_b satisfies the property (m). Indeed, the norm topology of ℓ^1 is finer than and compatible with $\sigma(\ell^1,\ell^\infty)$, therefore $(\ell^1,\sigma(\ell^1,\ell^\infty))$ is a Mazur space by Proposition 1.2.22. Further, for showing the property (m) of X_b, consider a $\sigma(\ell^1,\ell^\infty)$-bounded set B of ℓ^1, which is also $||\cdot||_1$-bounded. Let S be the unit ball of ℓ^1. Then there exists a positive number λ such that

$$B \subseteq \lambda S$$

Now S is $\sigma(\ell^1,c_o)$-compact by Alaoglu Bourbaki theorem (cf. Proposition 1.2.15) and $\sigma(\ell^1,\phi) \subseteq \sigma(\ell^1,c_o)$; therefore, S is $\sigma(\ell^1,\phi)$-compact. Consequently, S is $\sigma(\ell^1,\phi)$-closed and so

the $\sigma(\ell^1, \phi)$-closure of B is $||\cdot||_1$-bounded or $\sigma(\ell^1,\ell^\infty)$-bounded. Hence X_b satisfies the property (m) by Proposition 3.3.2.

Now since $\beta(\ell^\infty,\ell^1)|_\phi \equiv ||\cdot||_\infty$ and $(\phi,||\cdot||_\infty)^* = \ell^1$, we get

$$X_b \equiv (\ell^1, \sigma(\ell^1,\ell^\infty), \sigma(\ell^1,\phi))$$
$$\gamma\text{-}X_b \equiv (\phi, \beta(\phi,\ell^1) \ , \beta(\ell^\infty,\ell^1)|\phi)$$
$$\gamma^2\text{-}X_b \equiv (\ell^1, \beta(\ell^1,\phi) \ , \beta(\omega,\phi)|_{\ell^1})$$

Hence X_b is γ-semireflexive.

<u>Note</u>: In this chapter, from now onwards unless otherwise specified, we shall consider a quasinormal bi-l.c. TVS $X_b \equiv (X,T_1,T_2)$ such that (X,T_1) is a Mazur space. Observe that a quasinormal space satisfies the property (m) (cf. Proposition 3.3.3) and the remark following Proposition 3.3.5.

Let us begin with a characterization of γ-semireflexive spaces contained in

<u>Theorem</u> 3.3: Let $\langle X,X_\gamma^*\rangle$ be an M-dual system corresponding to a bi-l.c. TVS $X_b \equiv (X,T_1,T_2)$. Then X_b is γ-semireflexive if and only if every bounded and closed set in the space $(X,\sigma(X,X_\gamma^*))$ is compact.

<u>Proof</u>: By Proposition 3.5.6 and the remark following Proposition 3.5.8,

$$X_{21}^{**} = (X_2^*, \ \beta(X_1^*,X)|_{X_2^*})^*$$
$$= (X_\gamma^*, \ \beta(X_1^*,X)|_{X_\gamma^*})^*$$
$$= (X_\gamma^*, \ \beta(X_\gamma^*,X))^* = X_\gamma^{**}$$

Hence, X_b is γ-semireflexive

$\iff$ $J : X \to X_{21}^{**}$ is onto

$\iff$ $J : X \to X_{\gamma}^{**}$ is onto

$\iff$ $(X, \sigma(X, \sigma(X, X_{\gamma}^{*}))$ is a semireflexive l.c. TVS.

$\iff$ every closed and bounded set in $(X, \sigma(X, X_{\gamma}^{*}))$ is compact (cf. Proposition 1.2.26). This completes the proof.

As simple consequences of the above result, we have

<u>Corollary</u> 3.4: Under the hypothesis of Theorem 3.3, a bi-l.c. TVS X_b is γ-semireflexive if and only if every T_1-bounded set in X is $\sigma(X, X_{\gamma}^{*})$-relatively compact.

<u>Proof</u>: Invcking the proof of the Proposition 3.5.8, we have that the $\sigma(X, X_{\gamma}^{*})$-bounded sets and $\sigma(X, X_1^{*})$-bounded sets are the same. Therefore in view of Proposition 1.2.11, $\sigma(X, X_{\gamma}^{*})$-bounded and T_1-bounded sets are the same. Hence by Theorem 3.3, X_b is γ-semireflexive if and only if every T_1-bounded and $\sigma(X, X_{\gamma}^{*})$-closed set is $\sigma(X, X_{\gamma}^{*})$-compact, or equivalently , the $\sigma(X, X_{\gamma}^{*})$-closure of a T_1-bounded set is $\sigma(X, X_{\gamma}^{*})$-compact, that is, every T_1-bounded set is $\sigma(X, X_{\gamma}^{*})$-relatively compact.
This completes the proof.

<u>Corollary</u> 3.5: Under the hypothesis of Theorem 3.3 if X_b is γ-semireflexive, then $(X_{\gamma}^{*}, \beta(X_{\gamma}^{*}, X))$ is barrelled.

<u>Proof</u>: If X_b is γ-semireflexive, then $(X, \sigma(X, X_{\gamma}^{*}))$ is semi-reflexive. Therefore by Proposition 1.2.27, $(X_{\gamma}^{*}, \beta(X_{\gamma}^{*}, X))$ is barrelled.

Another consequence of Theorem 3.3, which is useful for our work on Schauder decomposition in the subsequent chapter, is contained in

Proposition 3.6: A bi-l.c. TVS $X_b \equiv (X,T_1,T_2)$ is saturated and γ-semireflexive if and only if (X,T_1) is semireflexive.

Proof: Let (X,T_1) be semireflexive. Then X_b is saturated by Proposition 3.5.4 and so $\langle X,X_\gamma^*\rangle$ is an M-dual system by Proposition 1.2.29 and a T_1-bounded set is $\sigma(X,X_1^*)$-relatively compact by a characterization of the semireflexive space (X,T_1). Hence X_b is γ-semireflexive by the Proposition 3.4.

Conversely, if X_b is γ-semireflexive and saturated, then $J:X \to X_1^{**}$ is onto, and so (X,T_1) is semireflexive. This establishes the result.

A variation of Corollary 3.4 is

Proposition 3.7: A bi-l.c. TVS $X_b \equiv (X,T_1,T_2)$ for which $\langle X,X_\gamma^*\rangle$ is an M-dual system, is γ-semireflexive if and only if every T_1-bounded and $\sigma(X,X_2^*)$-closed set is $\sigma(X,X_2^*)$-compact.

Proof: In view of Corollary 3.4, it is enough to show that the topologies $\sigma(X,X_\gamma^*)$ and $\sigma(X,X_2^*)$ coincide on T_1-bounded subsets of X. Let us therefore, consider a T_1-bounded set B, a net $\{x_\delta\}_{\delta\in\Lambda}$ and a point x in B such that $x_\delta \to x$ in $\sigma(X,X_2^*)$. Choose $\varepsilon > 0$ and an $f \in X_\gamma^*$. Then by Proposition 3.5.5, there exists a $g \in X_2^*$ such that

$$\sup_{x\in B} |f(x)-g(x)| < \frac{\varepsilon}{3} .$$

Also, we can select a $\delta_o \in \Lambda$ with

$$|g(x_\delta)-g(x_o)| < \frac{\varepsilon}{3}, \quad \forall \ \delta \geq \delta_o.$$

Hence, for $\delta \geq \delta_o$

$$|f(x_\delta)-f(x)| \leq |f(x_\delta)-g(x_\delta)|+|g(x_\delta)-g(x_o)|+|g(x_o)-f(x_o$$
$$< \varepsilon$$

$$\Longrightarrow x_\delta \to x \text{ in } \sigma(X,X_\gamma^*).$$

Thus,

$$\sigma(X,X_\gamma^*)|_B \subseteq \sigma(X,X_2^*)|_B .$$

The other inclusion is trivially true and hence the result follows.

Concerning the inheritance of γ-semireflexivity by subspaces of a bi-l.c. TVS, let us first observe that it is also a bi-l.c. TVS with respect to the induced topologies. Also, we have

<u>Proposition</u> 3.8: If $X_b \equiv (X,T_1,T_2)$ is quasinormal, then $Y_b \equiv (Y,T_1|_Y, T_2|_Y)$ is also quasinormal.

<u>Proof</u>: Let $\{y_\delta\}_{\delta\in\Lambda}$ be a net in Y and y in Y such that $y_\delta \to y$ relative to $T_2|_Y$. Consequently, $y_\delta \to y$ in (X,T_2) and so by the quasinormal character of X_b, there exists a family $\mathcal{D}_1$ of seminorms generating T_1 such that for each $p \in \mathcal{D}_1$, we can find q in $\mathcal{D}_1$ with

$$p(y) \leq \underline{\lim_{\delta}} \ q(y_\delta)$$

$$\text{or, } p|_Y(y) \leq \underline{\lim_{\delta}} \ q|_Y(y_\delta).$$

Since $\{p|_Y : p \in \mathcal{D}_1\}$ is a family of seminorms generating $T_1|_Y$, Y_b is quasinormal.

Next, we prove

Proposition 3.9: Let $X_b \equiv (X, T_1, T_2)$ be a bi-l.c. TVS for which $\langle X, X_\gamma^*\rangle$ is an M-dual system and Y be a T_2-closed subspace of X such that $(Y, T_1|_Y)$ is a Mazur space and $\langle Y, Y_\gamma^*\rangle$ is an M-dual system. Then Y_b is γ-semireflexive whenever X_b is γ-semireflexive.

Proof: Let B be a $T_1|_Y$-bounded and $\sigma(Y, Y_2^*)$-closed subset of Y, where Y_2^* is the topological dual of $(Y, T_2|_Y)$. Clearly, B is T_1-bounded and since by Proposition 1.2.31

$$\sigma(Y, Y_2^*) = \sigma(X, X_2^*)|_Y$$

it follows that there exists a $\sigma(X, X_2^*)$-closed subset B_1 of X such that

$$B = Y \cap B_1$$

Consequently, B is $\sigma(X, X_2^*)$-closed in X, for Y being T_2-closed and convex subspace of X, is $\sigma(X, X_2^*)$-closed. Therefore, by Corollary 3.7, B is $\sigma(X, X_2^*)$-compact. Hence B is $\sigma(Y, Y_2^*)$-compact which in turn implies the γ-semireflexivity of Y_B.

4. γ-Reflexivity:

Restricting the canonical map J, we introduce the notion of γ-reflexivity and establish a sufficient condition for γ-reflexivity. With the terminology of the preceding section, we introduce

Definition 4.1: A bi-l.c. TVS $X_b \equiv (X, T_1, T_2)$ with the property (m) and (X, T_1) as a Mazur space, is said to be γ-reflexive if it is γ-semireflexive and the canonical embedding J is a topological isomorphism in each of the following cases

(i) $J:(X,T_1) \to (X_{21}^{**}, \beta(X_{21}^{**}, X_2^*))$; and

(ii) $J:(X,T_2) \to (X_{21}^{**}, \beta(X_{22}^{**}, X_2^*)|_{X_{21}^{**}})$

As an example of a γ-reflexive bi-l.c. TVS, we consider

Example 4.2: Let $X_b \equiv (\ell^1, ||\cdot||_1, \sigma(\ell^1, \phi))$. Then $(\ell^1, ||\cdot||_1)$ is clearly a Mazur space with its unit ball $\sigma(\ell^1, \phi)$-closed. Thus X_b is normal and so satisfies the property (m). Further,

$$\gamma\text{-}X_b \equiv (\phi, \beta(\phi, \ell^1), \beta(\ell^\infty, \ell^1)|_\phi); \text{ and}$$
$$\gamma^2\text{-}X_b \equiv (\ell^1, \beta(\ell^1, \phi), \beta(\omega, \phi)|_{\ell^1}).$$

Hence X_b is γ-semireflexive. Also by Propositions 1.4.2 and 1.4.3, $\beta(\ell^1 \phi) = ||\cdot||_1$ and $\beta(\omega, \phi)|_{\ell^1} = \sigma(\ell^1, \phi)$. Therefore, the canonical embedding which is nothing but the identity map, is a topological isomorphism. Hence X_b is γ-reflexive bi-l.c.TVS.

An example of a γ-semireflexive bi-l.c. TVS which is not γ-reflexive is contained in

Example 4.3: Indeed, it is the space $X_b \equiv (\ell^1, \sigma(\ell^1, \ell^\infty), \sigma(\ell^1, \phi))$ of Example 3.2, which is γ-semireflexive bi-l.c. TVS. Here the canonical embedding which is the identity map from ℓ^1 to ℓ^1 is not $\sigma(\ell^1, \ell^\infty)$-$\beta(\ell^1, \phi)$ continuous since $\sigma(\ell^1, \ell^\infty) \subsetneqq ||\cdot||_1 = \beta(\ell^1, \phi)$. Thus it is not γ-reflexive.

A sufficient condition for γ-reflexivity is contained in

Proposition 4.4: A γ-semireflexive bi-l.c. TVS $X_b \equiv (X,T_1,T_2)$ is γ-reflexive if the spaces (X,T_1) and (X,T_2) are infrabarreled spaces.

Proof: In order to establish the result, we need show that J is a topological isomorphism from (i) (X,T_1) to $(X_{21}^{**},\beta(X_{21}^{**},X_2^*))$ and (ii) (X,T_2) to $(X_{21}^{**},\beta(X_{22}^{**},X_2^*)|_{X_{21}^{**}})$.

(i) For proving the continuity of J, let us consider a net $\{x_\delta\}$ in X such that $x_\delta \to 0$ in T_1 and let v be a $\beta(X_{21}^{**},X_2^*)$ neighbourhood of origin. Then there exists a $\beta(X_1^*,X)$ bounded set B in X_2^* such that $B^0 \subseteq v$. As (X,T_1) is infrabarrelled, B is equicontinuous and so there is a T_1-neighbourhood u of origin such that $B \subseteq u^0$. Also, there exists a δ_0 such that

$$x_\delta \in u, \qquad \forall\ \delta \geq \delta_0$$

$$\Longrightarrow \quad |f(x_\delta)| \leq 1, \qquad \forall\ f \in u^0 \text{ and } \delta \geq \delta_0.$$

$$\text{or} \quad |(J\,x_\delta)(f)| \leq 1,\ \forall\ f \in u^0 \text{ and } \delta \geq \delta_0$$

$$\Longrightarrow \quad Jx_\delta \in u^{00}, \qquad \forall\ \delta \geq \delta_0.$$

But $u^{00} \subseteq B^0 \subseteq v$. Therefore, $Jx_\delta \in v$, $\forall\ \delta \geq \delta_0$. As v is an arbitrary neighbourhood of origin in $\beta(X_{21}^{**},X_2^*)$, it follows that $Jx_\delta \to 0$, in $\beta(X_{21}^{**},X_2^*)$. Thus J is T_1-$\beta(X_{21}^{**},X_2^*)$ continuous.

For showing the continuity of J^{-1}, consider a net $\{y_\delta\}_{\delta\in\Lambda}$ in X_{21}^{**} such that $y_\delta \to 0$ in $\beta(X_{21}^{**},X_2^*)$. Write

$$x_\delta = J^{-1}(y_\delta),\ \delta \in \Lambda\ .$$

Then by the note following Proposition 3.5.7, there exists a family $\mathcal{D}_1$ of seminorms generating T_1 such that for each $p \in \mathcal{D}_1$ we get a q in $\mathcal{D}_1$ such that

$$p(x_\delta) \leq \sup \{ |f(x_\delta)| : f \in X_2^* \cap V_q^o \}$$

$$= \sup_{f \in B} |(J\, x_\delta)(f)| = p_B(y_\delta) ,$$

where $B = X_2^* \cap V_q^o$, is a $\beta(X_1^*, X)$-bounded set. Thus $p(x_\delta) \to 0$ and hence J^{-1} is continuous

(ii) Proceeding exactly in the same way, we can show that

$$J : (X, T_2) \to (X_{21}^{**}, \beta(X_{22}^{**}, X_2^*)|_{X_{21}^{**}})$$

is a topological isomorphism.

Thus (i) and (ii) are satisfied and so the bi-l.c. TVS X_b is γ-reflexive. This completes the proof.

<u>Remark</u>: Let us mention here that the infrabarrell character of the space (X, T_1) in the hypothesis of the above proposition is essential for the validity of the result; for otherwise, we have the Example 4.3, where the space $(\ell^1, \sigma(\ell^1, \ell^\infty))$ is not infrabarrelled.

Further, let us observe that the space (X, T_2) is infrabarrelled if the space X_b is γ-reflexive. However, restricting X_b further, we obtain the following characterization.

<u>Theorem 4.5</u>: A saturated γ-semireflexive bi-l.c. TVS $X_b \equiv (X, T_1, T_2)$ is γ-reflexive if and only if the spaces (X, T_1) and (X, T_2) are infrabarrelled.

<u>Proof</u>: In view of Proposition 4.3, we need prove the necessity part. So, let us assume that a saturated bi-l.c. TVS X_b is γ-reflexive. Then by Proposition 3.6 and also by hypothesis of γ-reflexivity $J:(X,T_1) \to (X_{11}^{**}, \beta(X_{11}^{**}, X_1^*))$ is a topological isomorphism. Hence (X,T_1) is reflexive and therefore infra-barrelled by Proposition 1.2.28.

Next, we show that (X,T_2) is infrabarrelled. Therefore, consider a $\beta(X_2^*,X)$-bounded set B in X_2^*. Then $B^o \cap X_{11}^{**}$ is a $\beta(X_{22}^{**},X_2^*)|_{X_{11}^{**}}$-neighbourhood of origin in X_{11}^{**}. Since $J:(X,T_2) \to (X_{11}^{**}, \beta(X_{22}^{**},X_2^*)|_{X_{11}^{**}})$ is a topological isomorphism, there exists a T_2-neighbourhood u of origin in X such that

$$J(u) \subset B^o \cap X_{11}^{**}$$

$$\Longrightarrow |f(x)| = |(Jx)\ (f)|$$

$$\leq 1, \quad \forall\ f \in B \ \text{ and } \ \forall\ x \in u.$$

$$\Longrightarrow B \subset u^o,$$

where the polars of B and u are considered in X_2^*. Hence B is an equicontinuous subset of X_2^* and therefore (X,T_2) is infrabarrelled This completes the proof.

CHAPTER - 5

CANONICAL BI-LOCALLY CONVEX SPACES

CONTENTS

Write

$$\mathcal{D}_{\bar{T}} = \{\bar{p} : p \in \mathcal{D}_T\} \text{ and}$$

$\mathcal{D}_{T^*} = \{p^* : p \in \mathcal{D}_T\}$.

We call the linear topology T^* generated by $\mathcal{D}_{T^*}$, the canonical topology on X.

Let us now start from a technical lemma.

Lemma 2.1 : Corresponding to an S.D. $\{M_n;P_n\}$ for a TVS (X,T), let p^* and $\bar{p}$ be as defined above. Then each of the functions p^* and $\bar{p}$ is an F-seminorm on X with

$$\bar{p}\left(\sum_{i=1}^{n} x_i\right) \leq \bar{p}(x),\ \forall\ n \geq 1,$$

where $x = \sum_{i\geq 1} x_i$. Also, each $\bar{p}$ in $\mathcal{D}_{\bar{T}}$ is T^*-lower semicontinuous. Further, if $\bar{T}$ (resp. T^*) is the linear topology generated by $\mathcal{D}_{\bar{T}}$ (resp. $\mathcal{D}_{T^*}$), then $T^* \subset \bar{T}$ and $T \subset \bar{T}$.

Proof : In order to show that p^* is an F-seminorm, we will prove $\alpha_n \to 0 \implies p^*(\alpha_n x) \to 0$, for each x in X; as the rest of the properties namely, (i) $p^*(x) \geq 0$ for each $x \in X$ and $p^*(0) = 0$; (ii) $p^*(x+y) \leq p^*(x) + p^*(y)\ \forall\ x,y \in X$; and (iii) $p^*(\alpha x) \leq p^*(x)$ $\forall\ |\alpha| \leq 1$, are clearly satisfied. Let us therefore consider $\alpha_n \to 0$ and $x \in X$. Then there exists an $M \in \mathbb{N}$ such that $|\alpha_n| \leq M$, $\forall\ n \geq 1$. By the definition of p^*, for each $x \in X$ and $\varepsilon > 0$ there exists $K \equiv K(x,\varepsilon,M)$ in $\mathbb{N}$ such that

$$\sum_{i\geq K+1} \frac{p(x_i)}{2^i} < \frac{\varepsilon}{2M}\ .$$

Hence,

$$p^*(\alpha_n x) = \sum_{i=1}^{K} \frac{p(\alpha_n x_i)}{2^i} + \sum_{i\geq K+1} \frac{p(\frac{\alpha_n}{M} \cdot Mx_i)}{2^i}$$

$$\leq \sum_{i=1}^{K} \frac{p(\alpha_n x_i)}{2^i} + \sum_{i\geq K+1} \frac{p(Mx_i)}{2^i}$$

$$< \sum_{i=1}^{K} \frac{p(\alpha_n x_i)}{2^i} + M \cdot \frac{\varepsilon}{2M} .$$

But for some $n_o \in \mathbb{N}$,

$$p(\alpha_n x_i) < \frac{\varepsilon}{2K} \text{ for } n \geq n_o,\ 1 \leq i \leq K.$$

Therefore

$$p^*(\alpha_n x) < \varepsilon,\ \forall\ n \geq n_o.$$

Consequently,

$p^*(\alpha_n x) \to 0$ and so it is an F-seminorm.

To show $\bar{p}$ is an F-seminorm, we observe that $\bar{p}$ satisfies (i), (ii) and (iii) immediately. To show $\bar{p}(\alpha_n x) \to 0$ for each $x \in X$ as $\alpha_n \to 0$, we observe that if A is T-bounded and $\alpha_n \to 0$ then for each $p \in \mathcal{D}_T$,

(*) $$\lim_{n\to\infty} \sup_{x\in A} p(\alpha_n x) = 0$$

[Indeed, if (*) is not true, then there exist an $\varepsilon > 0$ and a subsequence $\{\alpha_{n_i}\}$ such that

$$\sup_{x\in A} p(\alpha_{n_i} x) > \varepsilon \text{ for each } i \geq 1.$$

Consequently, there exists a sequence $\{x_i\}$ in A with $p(\alpha_{n_i} x_i) > \varepsilon$, $\forall\ i \geq 1$. This contradicts that A is T-bounded]. Now for $x \in X$, $\{\sum_{i=1}^{n} x_i : n \geq 1\}$ is T-bounded and therefore by (*)

$$\lim_{j\to\infty} \sup_{n} p(\alpha_j \sum_{i=1}^{n} x_i) = 0$$

$$\Longrightarrow \lim_{j\to\infty} \bar{p}(\alpha_j x) = 0.$$

Hence $\bar{p}$ is also an F-seminorm.

Concerning the second part of the lemma, we note that

$$\bar{p}(\sum_{i=1}^{n} x_i) = \sup_{1\leq m\leq n} p(\sum_{i=1}^{m} x_i) \leq \bar{p}(x),\ \forall\ n \geq 1.$$

Next, to show that each $\bar{p} \in \mathcal{D}_{\bar{T}}$ is T^*-lower semicontinuous, consider a net $\{x_\delta\}_{\delta\in\Lambda}$ and a point x in X with $x_\delta \to x$ in T^*. Then for each $p \in \mathcal{D}_T$,

$$p^*(x_\delta - x) = \sum_{i\geq 1} \frac{p(x_{\delta,i} - x_i)}{2^i}$$

$$\to 0,$$

where $x_{\delta,i} = P_i(x_\delta)$, $\delta \in \Lambda$, and $i \geq 1$. Hence for $p \in \mathcal{D}_T$,

$$p(x_{\delta,i} - x_i) \xrightarrow{\delta} 0,\ \forall\ i \geq 1,$$

$$\Longrightarrow p(S_n(x_\delta - x)) \xrightarrow{\delta} 0,\ \forall\ n \geq 1,$$

and so $S_n(x_\delta) \xrightarrow{\delta} S_n(x)$ in (X,T), $\forall\ n \geq 1$. Thus for each $\varepsilon > 0$, there exists $\delta_0 \equiv \delta_0(n,\varepsilon)$ such that

$$p(S_n(x_\delta)) \geq p(S_n(x)) - \varepsilon,\ \forall\ \delta \geq \delta_0.$$

Now

$$\bar{p}(x_\delta) \geq p(S_n(x_\delta))$$

$$\geq p(S_n(x)) - \varepsilon, \ \forall \ \delta \geq \delta_o$$

$$\Longrightarrow p(S_n(x) \leq \underline{\lim_{\delta}} \ \bar{p}(x_\delta)$$

$$\Longrightarrow \bar{p}(x) \quad \leq \underline{\lim_{\delta}} \ \bar{p}(x_\delta).$$

This shows that each $\bar{p} \in \mathcal{D}_{\bar{T}}$ is T^*-lower semicontinuous.

For showing $T^* \subseteq \bar{T}$, we note that the S.D. $\{M_n;P_n\}$ is monotone with respect to $\bar{T}$ and so

$$\bar{p}(x_i) \leq 2 \ \sup_n \bar{p}(\sum_{i=1}^{n} x_i) = 2 \ \bar{p}(x), \ \forall \ i \geq 1$$

$$\Longrightarrow p^*(x) = \sum_{i\geq 1} \frac{\bar{p}(x_i)}{2^i} \leq \sum_{i\geq 1} \frac{\bar{p}(x)}{2^{i-1}} = 2 \ \bar{p}(x)$$

$$\Longrightarrow T^* \subseteq \bar{T}.$$

Finally for the relation $T \subseteq \bar{T}$, we observe that for $x \in X$,

$$p(\sum_{i=1}^{n} x_i) \leq \bar{p}(x), \ \forall \ n \geq 1$$

$$\Longrightarrow p(x) = \lim_{n\to\infty} p(\sum_{i=1}^{n} x_i) \leq \bar{p}(x).$$

Thus the lemma is completely established.

We now state the main result of this section in the form of

<u>Theorem</u> 2.2 : If (X,T) is an l.c. TVS with S.D. $\{M_n;P_n\}$ then the corresponding functions p^* and $\bar{p}$ are seminorms and the

topologies T^* and $\bar{T}$ are locally convex topologies with $T^* \subset \bar{T}$ and $T \subset \bar{T}$. Hence for an l.c. TVS (X,T) with S.D. $\{M_n;P_n\}$, $X_b \equiv (X,\bar{T},T^*)$ is a normal bi-l.c. TVS.

Proof : Since each p in $\mathcal{P}_T$ is a seminorm on X, so are therefore $\bar{p}$ and p^* on X. Hence the topologies $\bar{T}$ and T^* are locally convex topologies on X. The rest of the theorem now follows immediately from Lemma 2.1.

Note : The bi-l.c. TVS $X_b \equiv (X,\bar{T},T^*)$ arising out of an l.c. TVS (X,T) with an S.D. $\{M_n;P_n\}$ as seen in the preceding theorem is termed as canonical bi-l.c. TVS.

As simple consequences, we derive

Corollary 2.3 : If $\{M_n;P_n\}$ is a monotone S.D. of an l.c. TVS (X,T), then (X,T,T^*) is a normal bi-l.c. TVS.

Proof : By monotonocity of $\{M_n;P_n\}$

$$p\left(\sum_{i=1}^{n} x_i\right) \leq p\left(\sum_{i=1}^{n+1} x_i\right),$$

for each $n \geq 1$ and x in X with $x = \sum_{i \geq 1} x_i$, $x_i \in M_i$, $i \geq 1$.

Hence

$$p(x) = \lim_{n\to\infty} p\left(\sum_{i=1}^{n} x_i\right) = \sup_n p\left(\sum_{i=1}^{n} x_i\right);$$

and therefore $T = \bar{T}$. Thus $X_b \equiv (X,T,T^*)$ is a normal bi-l.c. TVS by Theorem 2.2.

Corollary 2.4 : If $\{M_n;P_n\}$ is an e-S.D. of an l.c. TVS (X,T), then (X,T,T^*) is a quasinormal bi-l.c. TVS.

Proof: By Proposition 1.3.7, T is equivalent to $\bar{T}$. Hence for $p \in \mathcal{D}_T$, there exists $q \in \mathcal{D}_T$ such that

$$p(x) \leq \bar{p}(x) \leq q(x), \ \forall \ x \in X.$$

The above inequality along with T^*-lower semicontinuity of $\bar{p}$ immediately yield

$$p(x) \leq \underline{\lim}_{\delta} \ q(x_\delta)$$

for a net $\{x_\delta\}$ and x in X with $x_\delta \to x$ relative to the topology T^*. Consequently, (X,T,T^*) is quasinormal.

3. Shrinking And Boundedly Complete Schauder Decomposition :

This section is devoted to finding necessary and sufficient conditions for an S.D. to be shrinking or boundedly complete via mixed structure which is bifurcated into two subsections, viz, shrinking S.D. and Boundedly complete S.D. Throughout this section we denote by X_1^* and X_2^* the topological duals of $(X,\bar{T})$ and (X,T^*) respectively where $\bar{T}$ and T^* are locally convex topologies on an l.c. TVS (X,T) having an S.D. $\{M_n;P_n\}$ as introduced in the preceding section and consider the bi-l.c. TVS $X_b \equiv (X,\bar{T},T^*)$ with its γ-dual X_γ^*.

Shrinking Schauder decomposition :

Let us begin with

Proposition 3.1 : A $\bar{T}$-bounded net $\{y_\delta\}$ in X for which $P_i(y_\delta) \to 0$ in $(X,\bar{T})$, $\forall \ i \geq 1$ is $\sigma(X,X_1^*)$-convergent to zero if and only if $X_1^* \subset X_\gamma^*$.

<u>Proof</u> : For proving the necessity, let us consider an $f \in X_1^*$ and a net $\{y_\delta\}$ in X such that $y_\delta \xrightarrow{\gamma} 0$. Then $\{y_\delta\}$ is $\bar{T}$-bounded and $y_\delta \to 0$ in (X,T^*). Since $p(P_k x) = \bar{p}(P_k x)$, $p^*(y_\delta) = \sum_{i \geq 1} \frac{\bar{p}(P_i(y_\delta))}{2^i}$ and consequently $P_i(y_\delta) \to 0$ in $(X,\bar{T})$, $\forall\ i \geq 1$. So, $y_\delta \to 0$ in $\sigma(X,X_1^*)$ by hypothesis. Hence $f \in X_\gamma^*$.

Conversely, let $X_1^* \subset X_\gamma^*$ and $\{y_\delta\}_{\delta \in \Lambda}$ be $\bar{T}$-bounded and $P_i(y_\delta) \xrightarrow{\bar{T}} 0$, $\forall\ i \geq 1$. Now

$$\bar{p}(P_i x) \leq \bar{p}(S_i(x)) + \bar{p}(S_{i-1}(x))$$

$$\leq 2\ \bar{p}(x),\ \forall\ i \geq 1.$$

$$\Longrightarrow \bar{p}(P_i(y_\delta)) \leq M,\ \forall\ i \geq 1 \text{ and } \delta \in \Lambda,$$

for a certain positive real number M. Choose $\varepsilon > 0$. Then there exist $N \in \mathbb{N}$ and $\delta_0 \in \Lambda$ such that

$$\sum_{i \geq N+1} \frac{1}{2^i} \leq \frac{\varepsilon}{2M}$$

and

$$\bar{p}(P_i(y_\delta)) < \frac{\varepsilon}{2N},\ 1 \leq i \leq N,\ \delta \geq \delta_0.$$

Hence

$$p^*(y_\delta) < \varepsilon,\ \forall\ \delta \geq \delta_0.$$

Thus $y_\delta \xrightarrow{\gamma} 0$ and so $f(y_\delta) \to 0$, for each $f \in X_1^*$. This completes the proof.

<u>Corollary</u> 3.2 : Let $\{M_n;P_n\}$ be an e-S.D. of a Mazur space (X,T). Then $X_b \equiv (X,T,T^*)$ is saturated if and only if each T-bounded net $\{y_\delta\}$ for which $P_i(y_\delta) \to 0$ in (X,T) for $i \geq 1$, converges to zero relative to $\sigma(X,X^*)$.

Proof : Since T is equivalent to $\bar{T}$ by Proposition 1.3.7 and $X_\gamma^* \subset X_1^*$ by Proposition 3.5.1, the result follows immediately.

We now come to the main result of this subsection.

Theorem 3.3 : The S.D. $\{M_n;P_n\}$ of an l.c. TVS (X,T) is $\bar{T}$-shrinking at each $f \in X_\gamma^*$. Conversely, if $\{M_n;P_n\}$ is $\bar{T}$-shrinking at $f \in X_1^*$, then $f \in X_\gamma^*$.

Proof : We observe that $\{M_n;P_n\}$ is an e-S.D. for $(X,\bar{T})$. Indeed, from Lemma 2.1,

$$\bar{p}(S_n(x)) \leq \bar{p}(x), \ \forall\ n \geq 1,\ x \in X;$$

and

$$\begin{aligned}\bar{p}(S_n(x)-x) &= \sup_m p\left[S_m(S_n(x) - x)\right] \\ &= \sup_{m \geq n+1} p\left[S_n(x) - S_m(x)\right] \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

For proving the first statement, consider an $f \in X_\gamma^*$ and a $\bar{T}$-bounded set A in X. Then we need show

$$\lim_{n\to\infty} \sup_{x \in A} |\langle x, f - \sum_{i=1}^{n} P_i^*(f)\rangle| = 0,$$

or, equivalently, for a given $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$\sup_{x \in A} |f(x - S_n(x))| < \varepsilon, \ \forall\ n \geq N_0.$$

Let $p \in \mathcal{D}_T$ and choose a constant $M > 0$ such that

$$\bar{p}(x) \leq M, \ \forall\ x \in A.$$

Then for all $x \in A$,

$$p^*(x-S_n(x)) = \sum_{i\geq 1} \frac{\bar{p}(P_i(x-S_n(x)))}{2^i}$$

$$= \sum_{i\geq n+1} \frac{\bar{p}(P_i(x))}{2^i}$$

$$\leq \sum_{i\geq n+1} \frac{2 \sup_n \bar{p}(S_n(x))}{2^i} = \sum_{i\geq n+1} \frac{\bar{p}(x)}{2^{i-1}}$$

Hence

(*) $$\sup_{x\in A} p^*(x-S_n(x)) \leq \sum_{i\geq n+1} \frac{M}{2^{i-1}}, \ \forall\, n \geq 1.$$

Since the sequence $\{I-S_n\}$ is $\bar{T}$-$\bar{T}$ equicontinuous on X, the set $B = \bigcup_{n\geq 1} (I-S_n)(A)$ is also $\bar{T}$-bounded. Indeed, for $\bar{p} \in \mathcal{D}_{\bar{T}}$

$$\bar{p}(x-S_n(x)) \leq 2\, \bar{p}(x), \ \forall\, n \geq 1$$

$$\Longrightarrow \sup_{x\in A} \bar{p}(x-S_n(x)) \leq 2M, \ \forall\, n \geq 1$$

for some constant $M > 0$,

$$\Longrightarrow \sup_{x\in B} \bar{p}(x) \leq 2M.$$

Choose $\varepsilon > 0$. Then there exists $p^* \in \mathcal{D}_{T^*}$ and a positive number δ such that

(**) $$|f(x)| < \varepsilon, \text{ for } x \in B \cap \{x : p^*(x) < \delta\}$$

[For, on the contrary, there exists an $\varepsilon > 0$ and a bounded set B such that for each $p^* \in \mathcal{D}_{T^*}$ and $\delta = \frac{1}{n}$, there exists $x_n^\alpha \in B$ with $p^*(x_n^\alpha) < \frac{1}{n}$ and $|f(x_n^\alpha)| > \varepsilon$. This shows that we get a net

$\{x_n^\alpha\}$ defined on $\mathbb{N} \times \Lambda$ where Λ corresponds to neighbourhood system and is directed by the usual set inclusion such that $x_n^\alpha \xrightarrow{\gamma} 0$ but $f(x_n^\alpha) \not\to 0$, which is a contradiction to $f \in X_\gamma^*$].

Choose $N_o \in \mathbb{N}$ such that

$$\sum_{i \geq N_o+1} \frac{M}{2^{i-1}} < \delta .$$

Hence from (*)

$$\sup_{x \in A} p^*(x - S_n(x)) < \delta \ , \ n \geq N_o$$

Thus for $x \in A$, $y = x - S_n(x) \in B$ and

$$p^*(y) < \delta \ , \ \forall \ n \geq N_o.$$

Hence from (**)

$$|f(y)| < \varepsilon, \ \forall \ n \geq N_o$$

$$\Longrightarrow \sup_{x \in A} |f(x - S_n(x))| < \varepsilon, \ \forall \ n \geq N_o$$

Thus $\{M_n; P_n\}$ is $\bar{T}$-shrinking at $f \in X_\gamma^*$.

Conversely, assume that $\{M_n; P_n\}$ is $\bar{T}$-shrinking at an element f of X_1^*. For proving $f \in X_\gamma^*$, let us consider an arbitrary net $\{x_\delta\}_{\delta \in \Lambda}$ in X such that $x_\delta \xrightarrow{\gamma} 0$, that is, $\{x_\delta\}$ is $\bar{T}$-bounded and $x_\delta \to 0$ in (X, T^*). Since $x_\delta \to 0$ in (X, T^*) and $p^*(x_\delta) = \sum_{i \geq 1} \frac{\bar{p}(P_i(x_\delta))}{2^i}$, we find that

(**) $\qquad P_i(x_\delta) \to 0$ in $(X, \bar{T})$, $\forall \ i \geq 1$.

As $\{M_n; P_n\}$ is $\bar{T}$-shrinking at $f \in X_1^*$, from the $\bar{T}$-boundedness of

$\{x_\delta\}_{\delta \in \Lambda}$ we find that to $\varepsilon > 0$ there exists $n_o \equiv n_o(\varepsilon, \{x_\delta\})$ in $\mathbb{N}$ such that

$$\sup_\delta |f(x_\delta - \sum_{i=1}^{n} P_i(x_\delta))| < \frac{\varepsilon}{2}, \forall n \geq n_o,$$

$$\text{or,} \sup_\delta |f(\sum_{i \geq n_o+1} P_i(x_\delta))| < \frac{\varepsilon}{2}.$$

From (**) and the fact that $f \in X_1^*$, we have δ_o in Λ such that

$$|f(\sum_{i=1}^{n_o} P_i(x_\delta))| < \frac{\varepsilon}{2}, \forall \delta \geq \delta_o.$$

Consequently,

$$|f(x_\delta)| \leq |f(\sum_{i=1}^{n_o} P_i(x_\delta))| + |f(\sum_{i \geq n_o+1} P_i(x_\delta))|$$

$$< \varepsilon, \forall \delta \geq \delta_o$$

Hence $f \in X_\gamma^*$ and this establishes the result.

This result leads to the following characterizations of shrinking S.D. in a Mazur space.

Corollary 3.4 : An e-S.D. $\{M_n; P_n\}$ of a Mazur space (X,T) is shrinking if and only if the canonical bi-l.c. TVS $X_b \equiv (X, T, T^*)$ is saturated.

Proof : From our hypothesis $X_\gamma^* \subset X_1^*$ and $\bar{T}$ is equivalent to T. Since $\{M_n; P_n\}$ is shrinking at each $f \in X_1^*$, $X_1^* \subset X_\gamma^*$ by the Preceding result and consequently X_b is saturated.

Conversely, $\{M_n; P_n\}$ is shrinking at each $f \in X_\gamma^* = X_1^*$ and it is a shrinking S.D. and the result follows.

Now combining Corollary 3.2 and Corollary 3.4 we get

Proposition 3.5 : An e-S.D. $\{M_n;P_n\}$ of a Mazur space (X,T) is shrinking if and only if each T-bounded net $\{y_\delta\}$ for which $P_i(y_\delta) \to 0$ in (X,T) for $i \geq 1$, converges to origin relative to $\sigma(X,X^*)$.

Invoking the proof of Theorem 3.3, we observe that for $f \in X_\gamma^*$, the γ-dual of $X_b \equiv (X,\bar{T},T^*)$, $\{\sum_{i=1}^{n} P_i^*(f)\}$ converges to f relative to the topology of uniform convergence on $\bar{T}$-bounded sets in X. Indeed, we have a stronger result contained in

Proposition 3.6 : If $P_i^*(X_\gamma^*) = \{P_i^*(f): f \in X_\gamma^*\}$, $\forall\ i \geq 1$, then the sequence $\{P_i^*(X_\gamma^*);\ P_i^*\}$ is an e-S.D. of X_γ^* with respect to T_u, the topology of uniform convergence on $\bar{T}$-bounded subsets of X.

Proof : In view of first part, Theorem 3.3, we first show that $P_k^*(X_\gamma^*) \subset X_\gamma^*$ and each P_k^* is T_u-T_u continuous, $k \geq 1$. So, for given $k \geq 1$ consider $P_k^*(f)$ for $f \in X_\gamma^*$ and a net $\{x_\delta\}_{\delta \in \Lambda}$ in X such that $x_\delta \xrightarrow{\gamma} 0$ in $X_b \equiv (X,\bar{T},T^*)$. Then $P_k(x_\delta) \to 0$ in $(X,\bar{T})$ and hence $P_k(x_\delta) \xrightarrow{\gamma} 0$. Since $f \in X_\gamma^*$, we have

$$f(P_k(x_\delta)) \to 0$$

$$\text{or, } P_k^*(f)\ (x_\delta) \to 0.$$

This shows that $P_k^*(f) \in X_\gamma^*$.

For showing T_u-T_u continuity of P_i^*, $i \geq 1$, let $\{f_\delta\}_{\delta \in \Lambda}$ be a net in X_γ^* with $f_\delta \to 0$ in T_u and B be a $\bar{T}$-bounded set in X. Then $P_i(B)$ is also $\bar{T}$-bounded for each $i \geq 1$. Hence

$$\sup_{y \in P_i(B)} |f_\delta(y)| \to 0, \ \forall\ i \geq 1.$$

But,

$$\sup_{x \in B} |P_i^*(f_\delta)(x)| = \sup_{y \in P_i(B)} |f_\delta(y)|, \ \forall\ i \geq 1$$

$$\Longrightarrow P_i^*(f_\delta) \to 0 \text{ in } T_u.$$

Thus the desired continuity of P_i^* follows.

For T_u-T_u equicontinuity of $\{S_n^*\}$, observe that, $B_1 = \bigcup_{n \geq 1} S_n(B)$ is $\bar{T}$-bounded for each $\bar{T}$-bounded set B in X. Indeed,

$$\sup_{x \in B} \bar{p}(S_n(x)) \leq \sup_{x \in B} \bar{p}(x) < \infty, \ \forall\ n \geq 1$$

$$\Longrightarrow \sup_{x \in B} \bar{p}\Big(\bigcup_{n \geq 1} S_n(x)\Big) < \infty.$$

Now

$$\sup_{x \in B} |S_n^*(f)(x)| = \sup_{x \in B} |f(S_n(x))|$$
$$\leq \sup_{y \in B_1} |f(y)|$$

This completes the proof of the Proposition.

As an immediate consequence, we have

<u>Corollary</u> 3.7 : If $\{M_n; P_n\}$ is an e-S.D. of a Mazur space (X,T) and X_γ^* is the γ-dual of (X,T,T^*), then $\{P_n^*(X_1^*); P_n^*\}$ is an e-S.D. of X_γ^* with respect to $\beta(X_1^*,X)|_{X_\gamma^*}$.

<u>Proof</u> : We note that in this case $P_k^*(X_1^*) = P_k^*(X_\gamma^*)$. For, it clearly follows that $P_k^*(X_1^*) \subset X_\gamma^*$ and therefore $P_k^*(P_k^*(X_1^*)) = P_k^*(X_1^*) \subset P_k^*(X_\gamma^*)$. The other containment is trivial as $X_\gamma^* \subset X_1^*$

in this case. The result now follows immediately from the foregoing result.

The following result leads to another characterization of a shrinking S.D. in Mazur spaces.

<u>Proposition</u> 3.8 : For any linear topology S compatible with $\langle X, X_1^* \rangle$ and coarser than $\bar{T}$, if the $\bar{T}$-S.D. $\{M_n; P_n\}$ is S-uniform then $X_1^* \subset X_\gamma^*$. If $X_1^* \subset X_\gamma^*$ then $\{M_n; P_n\}$ is a $\sigma(X, X_1^*)$-uniform S.D. of $(X, \bar{T})$.

<u>Proof</u> : For proving the first, let $f \in X_1^*$ and $\{x_\delta\}_{\delta \in \Lambda}$ be a net in X with $x_\delta \xrightarrow{\gamma} 0$. Since $\{x_\delta\}_{\delta \in \Lambda}$ is $\bar{T}$-bounded, $\sum_{i=1}^{n} P_i(x_\delta) \to x_\delta$ in (X,S) as $n \to \infty$, uniformly in $\delta \in \Lambda$. Therefore, for given an $\varepsilon > 0$, there exists $N_o \equiv N_o(\varepsilon, f)$ in $\mathbb{N}$ such that

$$|f(\sum_{i \geq n+1} P_i(x_\delta))| < \frac{\varepsilon}{2}, \ \forall\, n \geq N_o \text{ and } \forall\, \delta \in \Lambda .$$

Further, $x_\delta \to 0$ in (X, T^*)

$$\Longrightarrow P_i(x_\delta) \to 0 \text{ in } (X, \bar{T})$$

$$\Longrightarrow |f(P_i x_\delta)| < \frac{\varepsilon}{2N_o}, \ \forall\, \delta > \delta_o \text{ and } 1 \leq i \leq N_o.$$

Consequently,

$$|f(x_\delta)| \leq \sum_{i=1}^{N_o} |f(P_i(x_\delta))| + |f(\sum_{i \geq N_o+1} P_i(x_\delta))|$$

$$\leq N_o \cdot \frac{\varepsilon}{2N_o} + \frac{\varepsilon}{2}, \ \forall\, \delta \geq \delta_o$$

$$= \varepsilon, \ \forall\, \delta \geq \delta_o.$$

Thus $f \in X_\gamma^*$ and $X_1^* \subset X_\gamma^*$.

To establish the later part, let us assume $X_1^* \subset X_\gamma^*$ and let B be a $\bar{T}$-bounded set. Then for $f \in X_1^*$, it follows by Theorem 3.3 that

$$\sup_{x \in B} |(f - \sum_{i=1}^{n} P_i^*(f))(x)| < \varepsilon, \ \forall\, n \geq N_o$$

for some integer $N_o \in \mathbb{N}$. Consequently, $\{M_n;P_n\}$ is $\sigma(X,X_1^*)$-uniform.

This result immediately yields

<u>Proposition</u> 3.9 : An e-S.D. $\{M_n;P_n\}$ of a Mazur space (X,T) is $\sigma(X,X^*)$-uniform if and only if $\{M_n;P_n\}$ is shrinking.

<u>Proof</u> : Follows immediately from the Preceding result.

The above result includes the following result of McArthur and Retherford (cf. [79], Theorem 3, p. 21o)

<u>Theorem</u> 3.10 : A weak S.D. $\{M_n;P_n\}$ for a Banach space X is weak-uniform if and only if $\{M_n;P_n\}$ is shrinking.

However, in this direction a better information is known; see for instance [65], Theorem 3.2.

For further discussion in this direction we need some preparations. Indeed, for an S.D. $\{M_n;P_n\}$ in an l.c. TVS (X,T) and the corresponding bi-l.c. TVS $X_b \equiv (X,\bar{T},T^*)$ with $X_1^* = (X,\bar{T})^*$, define

$$A_n = [\bigcup_{\substack{i=1 \\ i \neq n}}^{\infty} P_i^*(X_1^*)]$$

where $[\bigcup_{\substack{i=1 \\ i \neq n}}^{\infty} P_i^*(X_1^*)]$ denotes the $\beta(X_1^*,X)$-closure of the linear

span of $\bigcup_{\substack{i=1 \\ i \neq n}}^{\infty} P_i^*(X_1^*)$ in X_1^*.

Proposition 3.11 : If $(X,\bar{T})$ is a Mazur space, then $X_1^* \subset X_\gamma^*$ if for some n, A_n is $\sigma(X_1^*,X)$-closed. In other words, the bi-l.c. TVS X_b is saturated provided A_n is $\sigma(X_1^*,X)$-closed for some n.

Proof : From Proposition 3.5.2, X_γ^* is a $\beta(X_1^*,X)$-closed subspace of X_1^* and let A_{n_o} be $\sigma(X_1^*,X)$-closed for some $n_o \in \mathbb{N}$. Consider $f \in X_1^*$. Then

$$f = \sigma(X_1^*,X)\text{-} \lim_{n\to\infty} \sum_{i=1}^{n} P_i^*(f)$$

$$\Longrightarrow f-P_{n_o}^*(f) = \sigma(X_1^*,X) - \lim_{n\to\infty} \sum_{\substack{i=1 \\ i\neq n_o}}^{n} P_i^*(f)$$

$$\Longrightarrow f-P_{n_o}^*(f) \in A_{n_o}$$

$$\Longrightarrow f \in [\bigcup_{i=1}^{\infty} P_i^*(X_1^*)] .$$

Since $P_i^*(X_1^*) \subset X_1^*$, $\forall\, i \geq 1$, the result follows.

As an immediate consequence we have the following sufficient condition for an S.D. of a Mazur space to be shrinking.

Corollary 3.12 : If (X,T) is a Mazur space with e-S.D. $\{M_n;P_n\}$ and for some n, A_n is $\sigma(X^*,X)$-closed then $\{M_n;P_n\}$ is shrinking.

The converse statement of Proposition 3.11 is true in the following form.

Proposition 3.13 : If $X_1^* \subset X_\gamma^*$ then each A_n is $\sigma(X_1^*,X)$-closed.

Proof : Let $f \in \bar{A}_n$, the closure being w.r.t. $\sigma(X_1^*,X)$. Then there exists a net $\{f_\delta\}_{\delta \in \Lambda}$ in A_n such that $f_\delta \to f$ in $\sigma(X_1^*,X)$. Since $X_1^* \subset X_\gamma^*$, $\{M_n;P_n\}$ is shrinking at f and therefore

$$f = \beta(X_1^*,X) - \lim_{n\to\infty} \sum_{i=1}^{n} P_i^*(f).$$

By making use of $\sigma(X_1^*,X) - \sigma(X_1^*,X)$ continuity of P_n^* and the fact $P_n^*(f_\delta) = 0$, $\forall\, n \geq 1$ and $\forall\, \delta \in \Lambda$, we have $P_n^*(f) = 0$. [Indeed,

$$P_n^*(f_\delta) \to P_n^*(f) \text{ in } \sigma(X_1^*,X)$$

and since $f_\delta \in A_n$ for each $\delta \in \Lambda$, $P_n^*(f_\delta) = 0$]. Consequently, $f \in A_n$. This completes the proof.

As an immediate consequence, we have

Proposition 3.14 : If an e-S.D. $\{M_n;P_n\}$ of a Mazur space (X,T) is shrinking then each A_n is $\sigma(X_1^*,X)$-closed.

Boundedly complete Schauder decomposition :

The relationship between boundedly complete S.D. and the mixed structure is exhibited in

Theorem 3.15 : An S.D. $\{M_n;P_n\}$ of an l.c. TVS (X,T) is $\bar{T}$-boundedly complete if X_b is γ-sequentially complete. Conversely, if $\{M_n;P_n\}$ is $\bar{T}$-boundedly complete and each M_n is $\bar{T}$-complete (resp. $\bar{T}$-sequentially complete), then X_b is γ-complete (resp. γ-sequentially complete).

Proof : For the proof of the first part, let us consider a sequence $\{x_n\}$, $x_n \in M_n$, $n \geq 1$ such that $\{\sum_{i=1}^{n} x_i\}$ is $\bar{T}$-bounded in X. Write

$$y_n = \sum_{i=1}^{n} x_i, \; n \geq 1.$$

Then $\{y_n\}$ is $\bar{T}$-bounded and for each $p \in \mathcal{D}_T$, let $M = \sup_{n\geq 1} \bar{p}(y_n)$.
Further, $\{y_n\}$ is T^*-Cauchy sequence, for

$$\begin{aligned} p^*(y_n - y_m) &= \sum_{i=n+1}^{m} \frac{\bar{p}(x_i)}{2^i}, \; \forall\, m \geq n \\ &\leq \sum_{i=n+1}^{m} \frac{2 \sup_n \bar{p}(S_n x)}{2^i} \\ &= \sum_{i=n+1}^{m} \frac{M}{2^{i-1}} \\ &\to 0 \text{ as } n,m \to \infty. \end{aligned}$$

Hence $\{y_n\}$ is a γ-Cauchy sequence in $X_b \equiv (X, \bar{T}, T^*)$. Since X_b is γ-sequentially complete, there exists $z \in X$ such that $y_n \xrightarrow{\gamma} z$, where $z = \sum_{k\geq 1} P_k(z)$ in $(X, \bar{T})$. But for given n,

$$P_k(y_n) = \begin{cases} x_k & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

Since $y_n \to z$ in (X, T^*), we have

$$P_k(y_n) \to P_k(z), \; \forall\, k \geq 1 \text{ in } (X, \bar{T}).$$

Consequently, $P_k(z) = x_k$, $\forall\, k \geq 1$. Since $z = \sum_{k\geq 1} P_k(z)$ in $(X, \bar{T})$, the first part follows.

We prove the converse assertion for γ-completeness and similarly follows the γ-sequential part. Let us therefore, consider a γ-Cauchy net $\{x_\delta\}_{\delta \in \Lambda}$ in X_b. Then $\{P_k(x_\delta)\}$ is $\bar{T}$-Cauchy in M_k, $\forall\ k \geq 1$ such that $P_k(x_\delta) \to y_k$ in M_k relative to topology $\bar{T}$, for each $k \geq 1$. Consequently, for $p \in \mathcal{D}_T$ and $n \geq 1$,

$$\lim_{\delta} \bar{p}(S_n(x_\delta)) = \bar{p}\left(\sum_{i=1}^{n} y_i\right)$$

Now from the e-Schauder character of $\{M_n;P_n\}$ with respect to $\bar{T}$ and the boundedness of $\{x_\delta\}_{\delta \in \Lambda}$, there exists a positive constant M depending on p such that

$$(*) \qquad \bar{p}(S_n(x_\delta)) \leq M,\ \forall\ n \geq 1 \text{ and } \delta \in \Lambda.$$

Therefore,

$$\bar{p}\left(\sum_{i=1}^{n} y_i\right) \leq M,\ \forall\ n \geq 1.$$

Hence the sequence $\{\sum_{i=1}^{n} y_i\}$ is $\bar{T}$-bounded. Since $\{M_n;P_n\}$ is $\bar{T}$-boundedly complete, there exists $y \in X$ such that

$$\sum_{i=1}^{n} y_i \to y \text{ in } (X,\bar{T}).$$

$$\Longrightarrow P_k(y) = y_k,\ \forall\ k \geq 1.$$

The proof of this part will be disposed of showing that $x_\delta \xrightarrow{\gamma} y$. From (*), there is a positive constant M_1 such that

$$\bar{p}(P_k(x_\delta - y)) \leq M_1,\ \forall\ k \geq 1 \text{ and } \forall\ \delta \in \Lambda.$$

Also there exists a positive integer N_o such that

$$\sum_{k\geq N_o+1} \frac{1}{2^k} < \frac{\varepsilon}{2M_1}$$

Since $P_k(x_\delta) \to y_k$ in $(X,\bar{T})$ for each $k \geq 1$, for $\bar{p} \in \mathcal{D}_{\bar{T}}$ there exists δ_o such that

$$\bar{p}(P_k(x_\delta - y)) < \frac{\varepsilon}{2N_o}, \ \forall \ \delta \geq \delta_o \text{ and } 1 \leq k \leq N_o.$$

Hence,

$$p^*(x_\delta - y) = \sum_{k=1}^{N_o} \frac{\bar{p}(P_k(x_\delta - y))}{2^k} + \sum_{k\geq N_o+1} \frac{\bar{p}(P_k(x_\delta - y))}{2^k}$$

$$< \frac{\varepsilon}{2N_o} \cdot N_o + M_1 \cdot \frac{\varepsilon}{2M_1}$$

$$= \varepsilon, \ \forall \ \delta > \delta_o.$$

Consequently, $x_\delta \xrightarrow{\gamma} y$ and the proof is completed.

As an immediate consequence of this result, we derive

Proposition 3.16 : An e-S.D. $\{M_n;P_n\}$ of an l.c. TVS (X,T) is T-boundedly complete if $X_b \equiv (X,T,T^*)$ is γ-sequentially complete. Conversely, if X_b is γ-complete (resp. γ-sequentially complete) if $\{M_n;P_n\}$ is T-boundedly complete and each M_k is T-complete (resp. T-sequentially complete).

Proof : Since $\{M_n;P_n\}$ is e-S.D., T is equivalent to $\bar{T}$ and the result follows immediately.

4. Mixed Structure in Vector-valued Sequence Spaces :

To seek applications of the previous results in specific spaces, we confine to an arbitrary vector-valued sequence space

(VVSS) $\Lambda(X)$ introduced in the fourth section of Chapter 1. An obvious reason for considering the space $\Lambda(X)$ is that it always contains an S.D. $\{N_i;P_i\}$ relative to the topology $\sigma(\Lambda(X), \Lambda^{\times}(X^*))$; see for instance Theorem 1.4.8. Our basic results relating S.D. and the mixed structure essentially depend upon the fact that the S.D. in question should be an e-S.D. If we assume that the natural S.D. $\{N_i;P_i\}$ is e-Schauder, then $\Lambda^{\times}(X^*) = \Phi(X^*)$ by Theorem 1.4.8. Hence, in this case, the canonical topology coincides with $\sigma(\Lambda(X), \Lambda^{\times}(X^*))$ and it becomes a futile exercise to study the mixed structure of $\Lambda(X)$ relative to $\sigma(\Lambda(X), \Lambda^{\times}(X^*))$ and the canonical topology. Accordingly, let us investigate VVSS with suitable topologies and Schauder decompositions where we do not face a situation like the one mentioned above.

Thus, corresponding to a perfect sequence space λ and an l.c. TVS (X,T), let us recall the VVSS $\lambda(X)$ equipped with a solid topology $\mathfrak{J}$, and the spaces N_i, $i \geq 1$ defined in the fourth section of Chapter 1. Clearly each N_i is a subspace of $\lambda(X)$ with projection map P_i, $i \geq 1$ defined from $\lambda(X)$ to N_i with $P_i(\bar{x}) = \delta_i^{x_i}$, $i \geq 1$. In this section, we show that $\{N_i;P_i\}$ is an e-S.D. for $(\lambda(X), \mathfrak{J})$ that induces a mixed structure on $\lambda(X)$, and relate the γ-completeness of $(\lambda(X), \mathfrak{J})$ with the completeness of X.

Let us begin with a result on SVSS, namely,

<u>Lemma</u> 4.1 : Let λ be a sequence space and μ be a normal subspace of λ. Then $\sigma(\lambda,\mu)$- and $\eta(\lambda,\mu)$-compact (resp. relatively compact) sets in λ are the same.

Proof : It suffices to prove a $\sigma(\lambda,\mu)$ compact set is $\eta(\lambda,\mu)$-compact as the other implication is trivially true. So, let A be a $\sigma(\lambda,\mu)$-compact set. Then by Theorem 1.4.6, A is $\sigma(\lambda,\phi)$-compact and $\sigma(\lambda,\mu)$-sequential convergence coincides with that of $\sigma(\lambda,\phi)$ in A. Further, by Proposition 1.4.5, $\sigma(\lambda,\mu)$ and $\eta(\lambda,\mu)$ have the same convergent sequences; therefore, applying again the Theorem 1.4.6 (iv) ==> (i), we get $\eta(\lambda,\mu)$-compactness of A. The part for relative compactness follows analogously by applying the bracketed part of Theorem 1.4.6.

Proposition 4.2 : $\{N_i;P_i\}$ is an e-S.D. of $(\lambda(X),\mathcal{T})$.

Proof : Recalling the terminology from Section 4 of the Chapter 1, consider a set $A \in S$ and $\bar{x} = \{x_i\} \in \lambda(X)$. Then for $p \in \mathcal{D}_T$

$$Q_{A,p}(\bar{x} - \bar{x}^{(n)}) = \sup_{\bar{\beta}\in A} \sum_{i\geq n} p(x_i)\,|\beta_i|$$

$$\to 0 \text{ as } n \to \infty$$

by virtue of Proposition 1.4.3(iv) since the set $\{\{p(x_i)\beta_i\}:\bar{\beta}\in A\}$ is $\|\cdot\|_1$-compact as the image of the $\eta(\lambda^\times,\lambda)$-compact set A under the continuous map $R_p : (\lambda^\times, \eta(\lambda^\times,\lambda)) \to (\ell^1, \|\cdot\|_1)$, defined by $R_p(\bar{y}) = \{p(x_i)\; y_i\}$. Further,

$$Q_{A,p}(\bar{x}^{(n)}) = \sup_{\bar{\beta}\in A} \sum_{i=1}^{n} p(x_i)\;|\beta_i|$$

$$\leq Q_{A,p}(\bar{x}),\ \forall\, n \geq 1.$$

Hence $\{N_i;P_i\}$ is an e-S.D. of $(\lambda(X),\mathcal{T})$.

Remark : The S.D. character of $\{N_i;P_i\}$ for $(\lambda(X),\mathfrak{J})$ also follows from Proposition 1.4.9. However, in view of Proposition 4.2, we can define the canonical locally convex topology $\mathfrak{J}^*$ on $\lambda(X)$, generated by the family $\{\hat{Q}_{A,p} : A \in \mathcal{S},\ p \in \mathcal{D}_T\}$ of seminorms, where

$$\hat{Q}_{A,p}(\bar{x}) = \sum_{j\geq 1} \frac{\sup_{\bar{\beta}\in A} p(x_j)|\beta_j|}{2^j} .$$

Clearly, $(\lambda(X),\mathfrak{J},\mathfrak{J}^*)$ is a bi-locally convex space. Further, we have

Proposition 4.3 : If (X,T) is complete (resp. sequentially complete) then $(\lambda(X),\mathfrak{J},\mathfrak{J}^*)$ is γ-complete (resp. γ-sequentially complete).

Proof : Let $\{\bar{x}^\delta : \delta \in \Lambda\}$ be a γ-Cauchy net in $\lambda(X)$. Hence for given $\varepsilon > 0$, $A \in \mathcal{S}$ and $p \in \mathcal{D}_T$, we can find a constant $M > 0$ and a $\delta_o \in \Lambda$ such that

(i) $Q_{A,p}(\bar{x}^\delta) \leq M,\ \forall\ \delta \in \Lambda$

and

(ii) $\hat{Q}_{A,p}(\bar{x}^\delta - \bar{x}^\eta) < \varepsilon,\ \forall\ \delta,\ \eta \geq \delta_o$.

For fixed $j \in \mathbb{N}$, if we choose A in $\mathcal{S}$, containing the unit vector e^j, it follows from (ii) that

$$\sup_{\bar{\beta}\in A} \frac{p(x_j^\delta - x_j^\eta)\ |\beta_j|}{2^j} < \varepsilon,\ \forall\ \delta,\ \eta \geq \delta_o$$

$$\Longrightarrow \frac{p(x_j^\delta - x_j^\eta)}{2^j} < \varepsilon,\ \forall\ \delta,\ \eta \geq \delta_o .$$

Consequently, $\{x_j^\delta : \delta \in \Lambda\}$ is a Cauchy net in X. As $j \in \mathbb{N}$ is arbitrary and X is complete, we can find a sequence $\{x_j\} \subset X$ such that

$$x_j^\delta \to x_j \text{ in } (X,T), \forall j \geq 1.$$

Write $\bar{x} = \{x_j\}$. In order to show that $\bar{x} \in \lambda(X)$, consider a $\bar{\beta} \in \lambda^x$. Then there exists an $A \in S$ such that $\bar{\beta} \in A$ since S covers λ^x . Now for $p \in \mathcal{D}_T$, we have from (i),

$$\sup_{\bar{\beta} \in A} \sum_{i \geq 1} p(x_i^\delta) \ |\beta_i| \leq M, \forall \delta \in \Lambda$$

$$\Longrightarrow \sum_{i=1}^{n} p(x_i^\delta) \ |\beta_i| \leq M, \forall \delta \in \Lambda \text{ and } n \geq 1.$$

Taking the limit over δ , we get

$$\sum_{i=1}^{n} p(x_i) \ |\beta_i| \leq M, \forall n \geq 1$$

$$\Longrightarrow \sum_{i=1}^{\infty} p(x_i) \ |\beta_i| < \infty.$$

Since $\{\beta_i\}$ is an arbitrary member of λ^x, $\{p(x_i)\} \in \lambda^{xx}$. But $\lambda = \lambda^{xx}$, so $\bar{x} \in \lambda(X)$. Further, by the equicontinuity of $\{S_n\}$, $S_n = \sum_{i=1}^{n} P_i$, and the condition (i), we can find a constant $K > 0$ such that

$$Q_{A,p}(P_j(\bar{x}^\delta - \bar{x})) \leq K, \forall j \geq 1$$

Now for given $\varepsilon > 0$, choose $N \in \mathbb{N}$ with

$$\sum_{j>N} \frac{K}{2^j} < \frac{\varepsilon}{2} .$$

Consequently,

$$\hat{Q}_{A,p}(\bar{x}^{\delta} - \bar{x}) \leq \sum_{j=1}^{N} \frac{Q_{A,p}(P_j(\bar{x}^{\delta} - \bar{x}))}{2^j} + K \sum_{j>N} \frac{1}{2^j}$$

$$\to 0.$$

Thus $\bar{x}^{\delta} \xrightarrow{\gamma} \bar{x}$ and hence $(\lambda(X), \mathfrak{F}, \mathfrak{F}^*)$ is γ-complete. The proof for γ-sequentially completeness follows similarly.

In view of Proposition 3.16 an immediate consequence of the preceding result is

<u>Corollary</u> 4.4 : If (X,T) is sequentially complete then $\{N_i; P_i\}$ is a boundedly complete S.D.

As a partial converse to Proposition 4.3, we have

<u>Proposition</u> 4.5 : If $(\lambda(X), \mathfrak{F}, \mathfrak{F}^*)$ is γ-complete (resp. γ-sequentially complete) then (X,T) is a quasi-complete (resp. sequentially complete) space.

<u>Proof</u> : Let $\{x_\alpha : \alpha \in \Delta\}$ be a T-bounded Cauchy net in X. Consider the net $\{\delta_1^{x_\alpha} : \alpha \in \Delta\}$ in $\lambda(X)$, which is γ-Cauchy in $(\lambda(X), \mathfrak{F}, \mathfrak{F}^*)$, for if $A \in S$ and $p \in \mathcal{D}_T$, then there exist constants K and K_1 such that

$$p(x_\alpha) \leq K, \ \forall \ \alpha \in \Delta \quad \text{and} \ |\beta_1| \leq K_1, \ \forall \ \{\beta_i\} \in A,$$

the later being followed by the boundedness of $P_1(A)$ in $\mathbb{K}$.

Consequently,

$$Q_{A,p}(\delta_1^{\alpha}) = \sup_{\bar{\beta} \in A} p(x_\alpha) \ |\alpha_1| \leq KK_1$$

and

$$Q_{A,p}(\delta_1^{x_\alpha} - \delta_1^{x_\beta}) \leq K_1\, p(x_\alpha - x_\beta).$$

Since $(\lambda(X), \mathfrak{T}, \mathfrak{T}^*)$ in γ-complete, there exists an element $\bar{x}$ in $\lambda(X)$ such that $\delta_1^{x_\alpha} \xrightarrow{\gamma} \bar{x}$. Since $(\lambda(X), \mathfrak{T})$ is a GK-space by Proposition 1.4.9, $x_i = 0$ for $i \geq 2$ and $x_\alpha \to x_1$ in (X,T). Hence (X,T) is quasi-complete. Similarly, there follows the sequential completeness part of the result.

CHAPTER - 6

k-REFLEXIVITY AND γ-COMPLETION

CONTENTS :

1. Introduction :

In this chapter we continue our investigations on canonical bi-locally convex spaces, initiated in the preceding chapter. Indeed, in Section 2 we introduce the notion of a k-reflexive bi-l.c. TVS and characterize the class of such bi-l.c. TVS in terms of the subspaces forming the S.D. We study the notion of canonical γ-completion in the next section, which we identify as a vector-valued sequence space. In the last section of this chapter, we make use of the notions of Section 3 to characterize k-reflexivity and boundedly complete Schauder decomposition and establish a relationship between k- and γ-reflexivities of a canonical bi-l.c. TVS.

2. k-Reflexive Bi-Locally Convex Spaces :

For a Mazur space (X,T) with an e-S.D. $\{M_n;P_n\}$, it has been proved in the third section of the preceding chapter that $\{P_k^*(X_1^*);P_k^*\}$ is an e-S.D. of the γ-dual X_γ^* of the canonical bi-l.c. TVS $X_b \equiv (X,T,T^*)$, with respect to the topology $\beta(X_1^*,X)|_{X_\gamma^*}$ where $X_1^* = (X,T)^*$ (cf. Corollary 5.3.7). This fact is exploited to introduce the k-conjugate spaces as follows :

Definition 2.1 : Let (X,T) be a Mazur space with an e-S.D. $\{M_n;P_n\}$ and $X_b \equiv (X,T,T^*)$ be the corresponding canonical bi-l.c. TVS, where T^* is generated by $\mathcal{D}_{T^*}$ as defined in the beginning of the Section 2 of the preceding chapter. Then the canonical bi-l.c. TVS $(X_\gamma^*, \beta(X_1^*,X)|_{X_\gamma^*}, T_1^*)$ defined by the γ-dual X_γ^* of X_b

and its e-S.D. $\{P_k^*(X_1^*); P_k^*\}$ is termed as the first k-conjugate space of X_b and is denoted by k-X_b. In case $(X_\gamma^*, \beta(X_1^*,X)|_{X_\gamma^*})$ is a Mazur space, the canonical bi-l.c. TVS $(X_{\gamma\gamma}^{**}, \beta(X_{\gamma 1}^{**},X_\gamma^*)|_{X_{\gamma\gamma}^{**}}, T_2^*)$ where $X_{\gamma 1}^{**} = (X_\gamma^*, \beta(X_1^*,X)|_{X_\gamma^*})^*$, defined by the γ-dual $X_{\gamma\gamma}^{**}$ of the first k-conjugate space and its e-S.D. $\{P_k^{**}(X_{\gamma 1}^{**}); P_k^{**}\}$ is known as the second k-conjugate space and is denoted by k^2-X_b. Analytically, we have

$$X_b \equiv (X,T,T^*)$$
$$k\text{-}X_b \equiv (X_\gamma^*, \beta(X_1^*,X)|_{X_\gamma^*}, T_1^*) \text{ and}$$
$$k^2\text{-}X_b \equiv (X_{\gamma\gamma}^{**}, \beta(X_{\gamma 1}^{**},X_\gamma^*)|_{X_{\gamma\gamma}^{**}}, T_2^*)$$

where T^*, T_1^*, T_2^* are canonical topologies generated respectively by the family of seminorms

$$\mathcal{D}_{T^*} = \{p^* : p \in \mathcal{D}_T \text{ and } p^*(x) = \sum_{j\geq 1} \frac{p(P_j x)}{2^j} \text{ for } x \in X\};$$

$$\mathcal{D}_{T_1^*} = \{p_B^* : B \text{ varies over } T\text{-bounded subsets of } X$$
$$\text{and } p_B^*(f) = \sum_{j\geq 1} \frac{p_B(P_j^* f)}{2^j} \text{ for } f \in X_\gamma^*\}; \text{ and}$$

$$\mathcal{D}_{T_2^*} = \{p_{B^*}^* : B^* \text{ varies over } \beta(X_1^*,X)|_{X_\gamma^*}\text{-bounded subsets}$$
$$\text{of } X_\gamma^* \text{ and } p_{B^*}^*(F) = \sum_{j\geq 1} \frac{p_{B^*}(P_j^{**}(F))}{2^j} \text{ for } F \in X_{\gamma\gamma}^{**}\}.$$

Let us observe that $P_i^{**}(X_{\gamma\gamma}^{**}) = P_i^{**}(X_{\gamma 1}^{**})$, $\forall\, i \geq 1$ (cf. Corollary 5.3.7) and so we denote both these subspaces by

the common symbol $X^{**}_{\gamma\gamma i}$, $i \geq 1$ for the sake of convenience.

In view of k-conjugate spaces defined above, it is natural to enquire whether we can define an embedding from X to $X^{**}_{\gamma\gamma}$. In this direction, we prove Theorem 2.3. However, let us mention here that we consider throughout a Mazur space (X, T) with an e-S.D. $\{M_n; P_n\}$ such that the γ-dual X^*_γ of the canonical bi-l.c. TVS $X_b \equiv (X, T, T^*)$ is Mazur relative to $\beta(X^*_1, X)|_{X^*_\gamma}$ and J is the map defined from X to $X^{**}_{\gamma 1}$ by the relation

$$(Jx)(f) = f(x), \ \forall \ f \in X^*_\gamma.$$

Then we begin with the following simple

<u>Lemma</u> 2.2 : $JM_k \subset X^{**}_{\gamma\gamma k}$, for each $k \geq 1$.

<u>Proof</u> : Let us fix $k \geq 1$ and consider $x_k \in M_k$. Then we have to prove that $Jx_k \in X^{**}_{\gamma\gamma}$ and $P^{**}_k(Jx_k) = Jx_k$ on X^*_γ. For showing $Jx_k \in X^{**}_{\gamma\gamma}$, consider a net $\{f_\delta : \delta \in \Lambda\}$ in X^*_γ such that $f_\delta \xrightarrow{\gamma} 0$ in k-X_b. Therefore, $\{f_\delta\}$ is $\beta(X^*_1, X)$-bounded and

$$\sum_{j \geq 1} \frac{p_B(P^*_j f_\delta)}{2^j} \rightarrow 0$$

for each T-bounded set B of X.

Consequently,

$$(P^*_j f_\delta)(x) \rightarrow 0, \ \forall \ x \in X \text{ and } j \geq 1$$

$$(*) \qquad \Longrightarrow f_\delta(x_k) \rightarrow 0$$

$$\Longrightarrow (Jx_k)(f_\delta) \rightarrow 0$$

$$\Longrightarrow Jx_k \in X^{**}_{\gamma\gamma}.$$

For showing $P_k^{**}(Jx_k) = Jx_k$ on X_γ^*, consider an $f \in X_\gamma^*$. Then

$$(P_k^{**}(Jx_k))(f) = (Jx_k)(P_k^* f)$$
$$= (P_k^* f)(x_k)$$
$$= f(x_k) = (Jx_k)(f)$$

$$\Longrightarrow JM_k \subset X_{\gamma\gamma k}^{**}.$$

Restricting (X,T) further, we have

<u>Theorem</u> 2.3 : Let (X,T) be also infrabarrelled. Then J maps X into $X_{\gamma\gamma}^{**}$ and is a topological isomorphism from (X,T) onto $(JX,\ \beta(X_{\gamma 1}^{**}, X_\gamma^*)|_{JX})$ and also from (X,T^*) onto $(JX, T_2^*|_{JX})$.

<u>Proof</u> : In order to show $Jx \in X_{\gamma\gamma}^{**}$ for $x \in X$, consider a net $\{f_\delta : \delta \in \Lambda\}$ in X_γ^* such that $f_\delta \overset{\gamma}{\to} 0$ in k-X_b. Then $\{f_\delta\}$ is $\beta(X_1^*, X)$-bounded and by (*) in Lemma 2.2

$$(*) \qquad f_\delta(x_j) \to 0, \ \forall\ x_j \in M_j \text{ and } j \geq 1.$$

Since (X,T) is infrabarrelled, $\{f_\delta : \delta \in \Lambda\}$ is equicontinuous and so there exists a $p \in \mathcal{D}_T$ such that

$$(**) \qquad |f_\delta(x)| \leq p(x),\ \forall\ x \in X \text{ and } \delta \in \Lambda .$$

As $x = \sum_{j\geq 1} x_j$, $x_j \in M_j$, $j \geq 1$, for p as in (**), there exists $N_0 \in \mathbb{N}$ with

$$p\Big(\sum_{j \geq N_0+1} x_j\Big) < \frac{\varepsilon}{2}$$

Hence by (**)

$$|f_\delta(\sum_{j\geq N_o+1} x_j)| < \frac{\varepsilon}{2}, \ \forall \ \delta \in \Lambda.$$

Also, from (*) there exists $\delta_o \in \Lambda$ such that

$$|f_\delta(x_j)| < \frac{\varepsilon}{2N_o}, \text{ for } \delta > \delta_o \text{ with } j = 1,\ldots,N_o.$$

$$|f_\delta(x)| \leq |f_\delta(\sum_{j=1}^{N_o} x_j)| + |f_\delta(\sum_{j\geq N_o+1} x_j)|$$

$$< \varepsilon, \ \forall \ \delta > \delta_o.$$

Thus $(Jx)(f_\delta) = f_\delta(x) \to 0$. Hence $Jx \in X^{**}_{\gamma\gamma}$ and this completes the first part of the result.

Following the argument similar to the proof of part (i) of Proposition 4.4.4, this part can be easily established. Indeed, replacing $\beta(X^{**}_{21}, X^*_2)$ by $\beta(X^{**}_{\gamma 1}, X^*_\gamma)|_{JX}$, X^{**}_{21} by $X^{**}_{\gamma\gamma}$ and X^*_2 by X^*_γ in the proof of part (i) of Proposition 4.4.4, this part follows.

In order to show J is $T^*\text{-}T^*_2|_{JX}$ isomorphism consider a seminorm $p^*_{B^*} \in \mathcal{D}_{T^*_2}$ and $x \in X$. Now

$$p^*_{B^*}(Jx) = \sum_{j\geq 1} \frac{p_{B^*}(P^{**}_j(Jx))}{2^j}$$

By Lemma 2.2, $P^{**}_j(Jx) = Jx_j$, $j \geq 1$, and so

$$p^*_{B^*}(Jx) = \sum_{j\geq 1} \frac{p_{B^*}(Jx_j)}{2^j}$$

$$\leq \sum_{j\geq 1} \frac{M\, p(x_j)}{2^j}$$

for some seminorm p in $\mathcal{D}_T$. Indeed, since J is $T-\beta(X^{**}_{\gamma 1}, X^*_\gamma)|_{JX}$ continuous, for each p_{B^*} there exist a $p \in \mathcal{D}_T$ and an $M \in \mathbb{R}_+$ such that

$$p_{B^*}(Jx) \leq M\, p(x),\ \forall\, x \in X.$$

Thus $p^*_{B^*}(Jx) \leq M\, p^*(x)$ and therefore J is $T^* - T^*_2$ continuous.

For the continuity of J^{-1}, observe that X_b is quasinormal by Corollary 5.2.4. Therefore, following the note after Proposition 3.5.7, for each $p \in \mathcal{D}_T$,

$$p^*(x) = \sum_{j\geq 1} \frac{p(x_j)}{2^j}$$

$$\leq \sum_{j\geq 1} \frac{p_{B^*}(Jx_j)}{2^j}$$

for some $\beta(X^*_1, X)|_{X^*_\gamma}$-bounded subset B^* of X^*_γ. Indeed, since (X,T) is Mazur and $X_b \equiv (X,T,T^*)$ is quasinormal (cf. Corollary 5.2.4) by the note following Proposition 3.5.7, for each $p \in \mathcal{D}_T$ there exist $q \in \mathcal{D}_T$ such that

$$(2.4) \qquad p(x) \leq \sup\{|(Jx)(f)| : f \in X^*_\gamma \cap V^o_q\} = p_{B^*}(Jx)$$

where $B^* = X^*_\gamma \cap V^o_q$ is $\beta(X^*_1, X)|_{X^*_\gamma}$-bounded. Consequently, for $p^* \in \mathcal{D}_{T^*}$, we get B^* as above such that

$$p^*(x) \leq \sum_{j\geq 1} \frac{p_{B^*}(Jx_j)}{2^j} = p^*_{B^*}(Jx)$$

Thus J is $T^*-T^*_2|_{JX}$ topological isomorphism. Hence the result follows.

The above theorem leads us to

Definition 2.5 : Let (X,T) be a Mazur, infrabarrelled space with e-S.D. $\{M_n;P_n\}$ such that $\{X_\gamma^*, \beta(X_1^*,X)|_{X_\gamma^*})$ is a Mazur space. Then the corresponding canonical bi-l.c. TVS $X_b \equiv (X,T,T^*)$ is called k-reflexive if the embedding J as defined above from X to $X_{\gamma\gamma}^{**}$ is onto.

Caution : In this chapter, unless otherwise specified we shall consider an l.c. TVS (X,T) which is Mazur, infrabarrelled with e-S.D. $\{M_n;P_n\}$ such that $(X_\gamma^*, \beta(X_1^*,X)|_{X_\gamma^*})$ is a Mazur space.

For our next result, we need the following general result contained in

Lemma 2.6 : If $R : (X^*, \beta(X^*,X)) \to (Y^*, \beta(Y^*,Y))$ is a topological isomorphism, then so is its adjoint $R^* : (Y^{**}, \beta(Y^{**},Y^*)) \to (X^{**}, \beta(X^{**},X^*))$ where X^{**} and Y^{**} are respectively the topological duals of $(X^*, \beta(X^*,X))$ and $(Y^*, \beta(Y^*,Y))$.

Proof : By Proposition 1.2.30, $R^* : Y^{**} \to X^{**}$ is $\sigma(Y^{**},Y^*)$ - $\sigma(X^{**},X^*)$ continuous and hence it is $\beta(Y^{**},Y^*)$ - $\beta(X^{**},X^*)$ continuous.

We now show that R^* is one-to-one. Indeed,

$$R^*(y_1^{**}) = R^*(y_2^{**}) \text{ for } y_1^{**}, y_2^{**} \in Y^{**}$$

$$\Longrightarrow \langle R^*(y_1^{**}),f\rangle = \langle R^*(y_2^{**}),f\rangle, \ \forall\ f \in X^*.$$

$$\Longrightarrow \langle y_1^{**}, Rf\rangle = \langle y_2^{**}, Rf\rangle, \ \forall\ f \in X^*.$$

Since R is onto, each g in Y^* has the representation g = Rf for some $f \in X^*$. Hence

$$<y_1^{**},g> = <y_2^{**},g>, \ \forall \ g \in Y^*,$$

that is, $y_1^{**} = y_2^{**}$. Thus R^* is one-to-one.

Next, to show that R^* is onto, consider $x^{**} \in X^{**}$. Define a linear functional F on Y^* as follows

$$F(Rf) = x^{**}(f), \ \forall \ f \in X^*.$$

Also, F is $\beta(Y^*,Y)$-continuous, for if a net $\{y_\alpha^*\} \subset Y^*$, converges to zero in $\beta(Y^*,Y)$, we can find a net $\{x_\alpha^*\} \subset X^*$, with $y_\alpha^* = R\, x_\alpha^*$ such that $x_\alpha^* \to 0$ in $\beta(X^*,X)$. Then

$$F(y_\alpha^*) = F(Rx_\alpha^*) = x^{**}(x_\alpha^*).$$

$$\to 0.$$

Thus $F \in Y^{**}$. Further, $R^*(F) = x^{**}$. Hence R^* is onto.

In order to prove the $\beta(X^{**},X^*)-\beta(Y^{**},Y^*)$ continuity of $(R^*)^{-1}$, it is now enough to show

$$(R^*)^{-1} = (R^{-1})^*.$$

This is clearly satisfied for if $x^{**} \in X^{**}$, $y^* \in Y^*$

$$<(R^{-1})^*(x^{**}),y^*> = <x^{**},R^{-1}(y^*)>$$

But $x^{**} = R^*(y^{**})$ for some $y^{**} \in Y^{**}$, and hence

$$<(R^{-1})^*(x^{**}),y^*> = <R^*(y^{**}), R^{-1}(y^*)>$$

$$= <y^{**}, RR^{-1}(y^*)>$$

$$= <(R^*)^{-1}(x^{**}),y^*>$$

As the above equalities are true for arbitrary $x^{**} \in X^{**}$ and $y^* \in Y^*$ it follows that $(R^*)^{-1} = (R^{-1})^*$ and hence the result follows.

Next, we prove

<u>Proposition</u> 2.7 : If $\{M_n;P_n\}$, $\{P_n^*(X_1^*);P_n^*\}$ and $\{P_n^{**}(X_{\gamma\gamma}^{**});P_n^*\}$ are respectively e-S.D. of (X,T), $(X_\gamma^*, \beta(X_1^*,X)|_{X_\gamma^*})$ and $(X_{\gamma\gamma}^{**}, \beta(X_{\gamma 1}^{**}, X_\gamma^*)|_{X_{\gamma\gamma}^{**}})$, then for each $j \geq 1$ there exists a topological isomorphism

$$\phi_j : (M_j^{**}, \beta(M_j^{**}, M_j^*)) \to (P_j^{**}(X_{\gamma\gamma}^{**}), \beta(X_{\gamma 1}^{**}, X_\gamma^*)|_{P_j^{**}(X_{\gamma\gamma}^{**})})$$

such that $\phi_j|_{JM_j}$ is an identity mapping.

<u>Proof</u> : For proving the result we first show that $(M_j^*, \beta(M_j^*, M_j))$ is topologically isomorphic to $(P_j^*(X_1^*), \beta(X_1^*,X)|_{P_j^*(X_1^*)})$ for each $j \geq 1$. Fix $j \geq 1$ and define $R : M_j^* \to P_j^*(X_1^*)$ by

$$R(f) = P_j^*(\hat{f}), \quad f \in M_j^*$$

where $\hat{f}$ is the continuous extension of f from M_j to the whole space X. Clearly R is well defined and linear, for

$$R(f+g) = P_j^* (\widehat{f+g}) = P_j^*(\hat{f}+\hat{g}) = R(f) + R(g);$$

$$R(\alpha f) = P_j^*(\widehat{\alpha f}) = \alpha\, P_j^*(\hat{f}) = \alpha\, R(f),$$

where $f, g \in M_j^*$ and α is a scalar. Also R is one-to-one. For,

$$R(g) = 0 \implies P_j^*(\hat{g}) = 0$$

$$\implies (P_j^*\hat{g})(x) = 0, \ \forall\, x \in X$$

$$\Longrightarrow \hat{g}(x_j) = 0, \; \forall \; x_j \in M_j$$

$$\Longrightarrow g(x_j) = 0, \; \forall \; x_j \in M_j$$

$$\Longrightarrow g = 0.$$

Also R is clearly onto, and thus it is an algebraic isomorphism from M_j^* to $P_j^*(X_1^*)$.

For showing the $\beta(M_j^*,M_j) - \beta(X_1^*,X)|_{P_j^*(X_1^*)}$ continuity of R, consider a net $\{f_\delta : \delta \in \Lambda\}$ in M_j^* such that $f_\delta \to 0$ in $\beta(M_j^*,M_j)$. Let B be a $\sigma(X,X_1^*)$-bounded or equivalently T-bounded subset of X. Then

$$\begin{aligned} p_B(R(f_\delta)) &= p_B(P_j^*(\hat{f}_\delta)) \\ &= \sup_{x \in B} |\langle P_j^*(\hat{f}_\delta), x\rangle| \\ &= \sup_{x \in B} |\langle \hat{f}_\delta, P_j x\rangle| \\ &= \sup_{y \in P_j(B)} |\langle f_\delta, y\rangle| \\ &= p_{B_1}(f_\delta) \\ &\to 0 \end{aligned}$$

as $B_1 = P_j(B) \subset M_j$ is $T|_{M_j}$-bounded. Hence $R(f_\delta) \to 0$ in $\beta(X_1^*,X)|_{P_j^*(X_1^*)}$ and so the desired continuity of R follows.

For the continuity of R^{-1}, consider a net $\{P_j^*(f_\delta)\}$ in $P_j^*(X_1^*)$ such that $P_j^*(f_\delta) \to 0$ in $\beta(X_1^*,X)|_{P_j^*(X_1^*)}$. This in turn implies that $P_j^*(f_\delta) \to 0$ in $\beta(X_1^*,X)$. Let A be a $\sigma(M_j,M_j^*)$-bounded set in M_j. Then A is $\sigma(X,X_1^*)$-bounded by Proposition 1.2.31.

(iii) $JM_j \subset M_j^{**}$ and (iv) $S|_{JM_j}$ and $(R^*)^{-1}|_{JM_j}$ are identity mappings.

Indeed, the containment (i) follows from Lemma 2.2. For (ii) note that

$$(Jx)(P_j^* f) = (P_j^* f)(x)$$
$$= f(P_j x)$$

for each $x \in X$ and $f \in X_1^*$. Therefore, if a net $P_j^*(f_\delta) \to 0$ in $\beta(X_1^*, X)|_{P_j^*(X_1^*)}$, then $f_\delta(x_j) \to 0$ for each $x_j \in M_j$ and so $Jx_j \in (P_j^*(X_1^*))^*$.

For proving (iii) consider an element $x_j \in M_j$ and a net $\{f_\delta : \delta \in \Lambda\}$ in M_j^* such that $f_\delta \to 0$ in $\beta(M_j^*, M_j)$. Then $f_\delta(x_j) \to 0$ for each $x_j \in M_j$, and so $Jx_j(f_\delta) \to 0$. Hence $Jx_j \in M_j^{**}$.

For (iv) consider

$$(S(Jx_j))(f) = (P_j^{**}(\hat{J}x_j))(f)$$
$$= (Jx_j)(P_j^* f)$$
$$= f(x_j) = (Jx_j)(f)$$

for each $f \in X_\gamma^*$ and $x_j \in M_j$. Hence $S|_{JM_j}$ is an identity mapping.

For showing $(R^*)^{-1}|_{JM_j}$ is an identity mapping, we prove that $R^*|_{JM_j}$ is an identity. Therefore, for $f \in M_j^*$ and $x_j \in M_j$ consider

$$(R^*(Jx_j))(f) = (Jx_j)\ (Rf)$$
$$= (Jx_j)\ (P_j^*(\hat{f}))$$

$$= f(x_j)$$

$$= (Jx_j)(f)$$

This shows that $R^*|_{JM_j}$ is an identity mapping and (iv) follows.

Thus $\phi_j|_{JM_j}$ is an identity and the result is now completely proved.

Using Proposition 2.7, we characterize k-reflexive bi-l.c. TVS as follows.

<u>Proposition</u> 2.8 : If (X,T) is sequentially complete, then $X_b \equiv (X,T,T^*)$ is k-reflexive if and only if each M_k is reflexive.

<u>Proof</u> : First of all we prove that the following two statements are equivalent

(i) $J : X \to X^{**}_{\gamma\gamma}$ is onto

(ii) $J : M_j \to P_j^{**}(X^{**}_{\gamma\gamma})$ is onto for each $j \geq 1$.

(i) ==> (ii). Let us consider an $F \in P_j^{**}(X^{**}_{\gamma\gamma})$. Since $P_j^{**}(X^{**}_{\gamma\gamma}) \subset X^{**}_{\gamma\gamma}$, there exists an $x \in X$ such that $Jx = F$. Now

$$\begin{aligned}
Jx \in P_j^{**}(X^{**}_{\gamma\gamma}) &\implies P_j^{**}(Jx) = Jx \\
&\implies Jx\,(P_j^* f) = f(x),\ \forall\, f \in X^*_\gamma \\
&\implies f(P_j x) = f(x),\ \forall\, f \in X^*_\gamma \\
&\implies P_j x = x \\
&\implies x \in M_j.
\end{aligned}$$

Hence (ii) follows.

(ii) ==> (i). Consider an $F \in X_{\gamma\gamma}^{**}$. Then there exist $x_j \in M_j$ such that $Jx_j = P_j^{**}(F)$, $j \geq 1$. Consequently,

$$F = \sum_{j\geq 1} P_j^{**}(F) \doteq \sum_{j\geq 1} Jx_j,$$

where the convergence of the series is considered relative to $\beta(X_{\gamma 1}^{**}, X_{\gamma}^{*})|_{X_{\gamma\gamma}^{**}}$. By the inequality (2.4) in the proof of the Theorem 2.3, it follows that

$$p(\sum_{j=m}^{n} x_j) \leq p_{B^*}(\sum_{j=m}^{n} Jx_j)$$

for all $n \geq m$. Consequently, $\{\sum_{j=1}^{n} x_j : n \geq 1\}$ is T-Cauchy and therefore there exists an $x \in X$ such that

$$x = \sum_{j\geq 1} x_j$$

$$\Longrightarrow Jx = \sum_{j\geq 1} Jx_j$$

Thus $F = Jx$, for some $x \in X$. Hence J maps X onto $X_{\gamma\gamma}^{**}$.

Now

X_b is k-reflexive

$\Longrightarrow J : X \to X_{\gamma\gamma}^{**}$ is a T-$\beta(X_{\gamma 1}^{**}, X_{\gamma}^{*})|_{X_{\gamma\gamma}^{**}}$-topological isomorphism

$\Longrightarrow J : M_j \to X_{\gamma\gamma j}^{**}$ is a $T|_{M_j}$ - $\beta(X_{\gamma 1}^{**}, X_{\gamma}^{*})|_{X_{\gamma\gamma j}^{**}}$ -topologic[al] isomorphism.

Since $JM_j = X_{\gamma\gamma j}^{**} = M_j^{**}$ by the preceding result, $J : M_j \to M_j^{**}$ is a $T|_{M_j}$ - $\beta(M_j^{**}, M_j^{*})$-topological isomorphism and hence M_j is reflexive for each $j \geq 1$.

Conversely, if each M_j is reflexive, then the map J maps X onto $X^{**}_{\gamma\gamma}$ by the above arguments. Hence X_b is k-reflexive

3. Canonical γ-Completion :

In this section we construct a VVSS equipped with two locally convex topologies corresponding to a canonical bi-l.c. TVS, that behaves like a γ-completion of a bi-l.c. TVS in the sense of the following

Definition 3.1 : Let $X_b \equiv (X, T_1, T_2)$ be a bi-l.c. TVS. If there exists a normal, γ-complete bi-l.c. TVS $\tilde{X}_b \equiv (\tilde{X}, \tau, \tau^*)$ containing X_b as a γ-dense subspace, then $\tilde{X}_b$ is known as a γ-completion of X_b. We call the γ-completion of a canonical bi-l.c. TVS as the canonical γ-completion.

Recalling the map $J : X \to X^{**}_{\gamma\gamma}$ as well as the restriction on the space (X, T, T^*) and its γ-dual from the preceding section, namely, (X,T) is a Mazur, infrabarrelled space with an e-S.D. $\{M_n; P_n\}$ such that the γ-dual X^*_γ of the corresponding canonical bi-l.c. TVS is Mazur relative to $\beta(X^*_1, X)|_{X^*_\gamma}$, we first prove

Proposition 3.2 : For a given $x_j \in M_j$, $j \geq 1$, the series $\sum_{j\geq 1} Jx_j$ is $\sigma(X^{**}_{\gamma 1}, X^*_\gamma)$-convergent provided $\{\sum_{j=1}^{n} Jx_j : n \geq 1\}$ is $\beta(X^*_1, X)|_{X^*_\gamma}$ equicontinuous.

Proof : Write $G_n = \sum_{j=1}^{n} Jx_j$, $n \geq 1$. Since $\{G_n : n \geq 1\}$ is $\beta(X^*_1, X)|_{X^*_\gamma}$-equicontinuous, there exists a T-bounded set B in X such that

$$(*) \qquad |\langle f, G_n\rangle| \leq p_B(f), \ \forall\, n \geq 1 \text{ and } f \in X^*_\gamma.$$

Let $g \in sp \bigcup_j P_j^*(X_1^*) = Y$. Then

$$g = \sum_{j=1}^{m} P_j^*(f_j)$$

Now

$$G_n(g) = (\sum_{j=1}^{n} Jx_j)(\sum_{j=1}^{m} P_j^*(f_j))$$

$$= \sum_{i=1}^{m} f_i(x_i), \quad \forall \quad n \geq m$$

$\Longrightarrow G_n(g)$ converges for all $g \in Y$.

Define a linear map $F:Y \to \mathbb{K}$ as follows

(**) $\qquad F(g) = \lim_{n\to\infty} G_n(g), \quad \forall\ g \in Y$

By (*)

$$|\langle g,F\rangle| \leq p_B(g), \quad \forall\ g \in Y$$

$\Longrightarrow F$ is $\beta(X_1^*,X)|_Y$-continuous.

Applying Hahn-Banach theorem, we can extend F to a $\beta(X_1^*,X)|_{X_\gamma^*}$-continuous linear $\hat{F}$ on X_γ^*.

For showing $\sigma(X_{\gamma 1}^{**},X_\gamma^*)$-convergence of $\sum_{j\geq 1} Jx_j$, consider $f \in X_\gamma^*$. Then

$$f = \sum_{j\geq 1} P_j^* f,$$

where the series converges relative to $\beta(X_1^*,X)|_{X_\gamma^*}$. Thus for B in (*), there exists n_o in $\mathbb{N}$ such that

$$p_B(f - \sum_{i=1}^{n_o} P_i^* f) < \frac{\varepsilon}{3}.$$

Also by (**), for $g \in Y$ there exists m_o in $\mathbb{N}$ such that

$$|\hat{F}(\sum_{j=1}^{n_o} P_j^* f) - G_n(\sum_{j=1}^{n_o} P_j^* f)| < \frac{\varepsilon}{3}, \quad \forall\, n \geq m_o$$

Consequently,

$$\begin{aligned}|\hat{F}(f)-G_n(f)| &\leq |G_n(f-\sum_{i=1}^{n_o} P_i^* f)| + |(G_n-\hat{F})(\sum_{j=1}^{n_o} P_j^* f)| \\ &\quad + |\hat{F}(\sum_{j=1}^{n_o} P_j^* f-f)| \\ &\leq p_B(f-\sum_{j=1}^{n_o} P_j^* f) + |(G_n-\hat{F})(\sum_{j=1}^{n_o} P_j^* f)| \\ &\quad + p_B(\sum_{j=1}^{n_o} P_j^* f-f) \\ &\leq \varepsilon, \qquad \forall\, n \geq m_o.\end{aligned}$$

Thus $\sum_{j\geq 1} Jx_j$ is $\sigma(X_{\gamma 1}^{**}, X_\gamma^*)$-convergent.

Note: Let us note that if $(X_\gamma^*, \beta(X_1^*, X)|_{X_\gamma^*})$ is infrabarrelled, then the convergence of the series $\sum_{j\geq 1} Jx_j$ in the topology $\sigma(X_{\gamma 1}^{**}, X_\gamma^*)$ implies the $\beta(X_1^*, X)|_{X_\gamma^*}$-equicontinuity of $\{\sum_{j=1}^{n} Jx_j : n \geq 1\}$; for in this case $(X_\gamma^*, \beta(X_1^*, X)|_{X_\gamma^*})$, being complete, is a barrelled space (cf. Proposition 1.2.20). Consequently, $\{\sum_{j=1}^{n} Jx_j : n\geq 1\}$ is $\beta(X_{\gamma 1}^{**}, X_\gamma^*)$-bounded.

This fact yields the construction of a VVSS equipped with two locally convex topologies in the following

Definition 3.3: In addition to our earlier restrictions, let us also assume that $(X_\gamma^*, \beta(X_1^*, X)|_{X_\gamma^*})$ is infrabarrelled and set

$$\tilde{X} = \{\{Jx_j\} : \sum_{j\geq 1} Jx_j \text{ converges in } \sigma(X_{\gamma 1}^{**}, X_\gamma^*)\}.$$

Then $\tilde{X}$ is a vector space with respect to usual pointwise addition and scalar multiplication. Further, it can be equipped with two locally convex topologies τ and τ^* defined respectively by the families of seminorms

$$\mathcal{D}_\tau = \{q_{B^*} : B^* \text{ varies over } \beta(X_1^*, X)|_{X_\gamma^*}\text{-bounded subsets of } X_\gamma^*$$
$$\text{and } q_{B^*}(\{Jx_j\}) = \sup_n p_{B^*}\left(\sum_{j=1}^{n} Jx_j\right) \text{ for } \{Jx_j\} \in \tilde{X}\}; \text{ ar}$$

$$\mathcal{D}_{\tau^*} = \{\hat{q}_{B^*} : B^* \text{ varies over } \beta(X_1^*, X)|_{X_\gamma^*}\text{-bounded subsets of } X_\gamma^*$$
$$\text{and } \hat{q}_{B^*}(\{Jx_j\}) = \sum_{j\geq 1} \frac{p_{B^*}(Jx_j)}{2^j} \text{ for } \{Jx_j\} \in \tilde{X}\}$$

Returning to the triplet $\tilde{X}_b \equiv (\tilde{X}, \tau, \tau^*)$, we have the following result concerning its basic structural properties.

Proposition 3.4: The VVSS $(\tilde{X}, \tau)$ and $(\tilde{X}, \tau^*)$ are GK-spaces.

Proof: Let $\{F_\delta\}_{\delta \in \Lambda}$ be a net and F a point in $\tilde{X}$ with $F = \{Jx_j^\delta\}$, $\delta \in \Lambda$ and $F = \{Jx_j\}$ such that $F_\delta \to F$ in $(\tilde{X}, \tau)$. Then for each $\beta(X_1^*, X)|_{X_\gamma^*}$-bounded subset B^* of X_γ^*

$$p_{B^*}(Jx_j^\delta - Jx_j) \leq 2 \sup_n p_{B^*}\left(\sum_{j=1}^{n} (Jx_j^\delta - Jx_j)\right)$$
$$= 2\, q_{B^*}(F_\delta - F)$$
$$\to 0$$

Consequently, $Jx_j^\delta \to Jx_j$ in $\beta(X_{\gamma 1}^{**}, X_\gamma^*)$ for each $j \geq 1$ and hence $(\tilde{X}, \tau)$ is a GK-space

Also $F_\delta \to F$ in τ^*

$$\Longrightarrow \sum_{j\geq 1} \frac{p_{B^*}(Jx_j^\delta - Jx_j)}{2^j} \xrightarrow{\delta} 0$$

$$\Longrightarrow p_{B^*}(Jx_j^\delta - Jx_j) \xrightarrow{\delta} 0 \text{ for each } j \geq 1.$$

Hence $(\tilde{X},\tau^*)$ is a GK-space.

Note: If $N_i = \{\delta_i^{Jx} : x \in X\}$, $i \geq 1$, $\{N_i\}$ is clearly an S.D. for $(\tilde{X},\tau^*)$. Also it is an S.D. for $(\tilde{X},\sigma(\tilde{X},\phi(X_\gamma^*)))$. However, it would be interesting to investigate the form of the generalized Köthe dual of $\tilde{X}$ and establish relationship with its topological duals (cf [52]) so as to have an insight of the various structural properties of the space $\tilde{X}$ and the role played by $\{N_i\}$ in view of the results of [54] .

Proposition 3.5: $\tilde{X}_b = (\tilde{X},\tau,\tau^*)$ is a normal bi-l.c. TVS.

Proof: Let us first show $\tilde{X}_b$ is a bi-l.c. TVS. Therefore, for a $\beta(X_1^*,X)|_{X_\gamma^*}$-bounded subset B^* of X_γ^* and $\{Jx_j\} \in \tilde{X}$, consider

$$\begin{aligned}
\hat{q}_{B^*}(\{Jx_j\}) &= \sum_{j\geq 1} \frac{p_{B^*}(Jx_j)}{2^j} \\
&\leq \sum_{j\geq 1} \frac{2 \sup_n p_{B^*}(\Sigma_{j=1}^n Jx_j)}{2^j} \\
&\leq q_{B^*}(\{Jx_j\}) \sum_{j\geq 1} \frac{1}{2^{j-1}} \\
&= 2\, q_{B^*}(\{Jx_j\})
\end{aligned}$$

Consequently , $\tau^* \subset \tau$ and $\tilde{X}_b$ is a bi-l.c. TVS.

For showing the normal character of $\tilde{X}_b$, let us take a net $\{F_\delta\}_{\delta\in\Lambda}$ and F in $\tilde{X}_b$ with $F_\delta = \{Jx_j^\delta\}$ $\forall\, \delta \in \Lambda$ and $F = \{Jx_j\}$ such that $F_\delta \to F$ in $(\tilde{X},\tau^*)$. Hence by Proposition 3.4, $p_{B^*}(Jx_j^\delta) \to p_{B^*}(Jx_j)$ for each $\beta(X_1^*,X)|_{X_\gamma^*}$-bounded subset B^* of X_γ^* and $j \geq 1$. We now fix a $\beta(X_1^*,X)|_{X_\gamma^*}$-bounded set B^* in X_γ^*. Then for $f \in B^*$ and $n \in \mathbb{N}$

$$\left|\sum_{j=1}^{n}(Jx_j)(f)\right| = \left|\sum_{j=1}^{n}(\lim_{\delta}(Jx_j^{\delta}))\ (f)\right|$$
$$= \lim_{\delta}\left|\sum_{j=1}^{n}J(x_j^{\delta})(f)\right|$$
$$\leq \underline{\lim}_{\delta}\ p_{B^*}\left(\sum_{j=1}^{n}J(x_j^{\delta})\right)$$
$$= \underline{\lim}_{\delta}\ q_{B^*}(F_{\delta})$$

Since the right hand side is independent of $f \in B^*$ and $n \in \mathbb{N}$, we get

$$q_{B^*}(F) \leq \underline{\lim}_{\delta}\ q_{B^*}(F_{\delta}).$$

Thus $\tilde{X}_b$ is a normal bi-l.c. TVS.

Note: Since $\tau^* \subset \tau$, let us mention here that the GK-character of $(\tilde{X},\tau)$ also follows from the GK-character of $(\tilde{X},\tau^*)$.

Proposition 3.6: If (X,T) is complete, $\tilde{X}_b$ is a γ-complete bi-l.c. TVS.

Proof: To prove the γ-completeness of $\tilde{X}_b \equiv (\tilde{X},\tau,\tau^*)$, consider a γ-Cauchy net $\{F_{\delta}: \delta \in \Lambda\}$ in $\tilde{X}_b$ where $F_{\delta} = \{Jx_j^{\delta}\}$. Then for a given $\beta(X_1^*,X)|_{X_{\gamma}^*}$ - bounded set B^* and $\varepsilon > 0$ there exists a positive constant M depending on B^* and a $\delta_o \in \Lambda$ such that

$$(*) \qquad q_{B^*}(F_{\delta}) \leq M, \quad \forall\, \delta \in \Lambda;$$

$$(**) \qquad \hat{q}_{B^*}(F_{\delta}-F_{\eta}) < \varepsilon, \quad \forall\, \delta,\eta \geq \delta_o.$$

Consequently, from (**) $\{Jx_j^{\delta}: \delta \in \Lambda\}$ is a Cauchy net in JM_j for each $j \geq 1$. Now, by the completeness of each $(M_j, T|_{M_j})$ and the topological isomorphic character of J from (X,T) to $(JX, \beta(X_{\gamma 1}^{**},X_{\gamma}^*)|_{JX})$, the completeness of $(JM_j, \beta(X_{\gamma 1}^{**},X_{\gamma}^*)|_{JM_j})$ follows. Hence there exist $x_j \in M_j$, $j \geq 1$ such that

$$\{Jx_j^{\delta}\} \to Jx_j, \text{ in } \beta(X_{\gamma 1}^{**},X_{\gamma}^*)|_{JM_j} \text{ and } j \geq 1.$$

In order to dispose of the proof completely, we need show that

(a) $F = \{Jx_j\} \in \tilde{X}$; and

(b) $F_\delta \xrightarrow{\gamma} F$ in $\tilde{X}_b$.

For proving (a) let us consider a $\beta(X_1^*, X)|_{X_\gamma^*}$-bounded subset B^* of X_γ^*. Then for $n \in \mathbb{N}$ and $f \in B^*$

$$
\begin{aligned}
\left|\sum_{j=1}^{n} (Jx_j)(f)\right| &\leq \lim_\delta \left|\sum_{j=1}^{n} (Jx_j^\delta)(f)\right| \\
&\leq \underline{\lim}_\delta \, p_{B^*}\left(\sum_{j=1}^{n} Jx_j^\delta\right) \\
&\leq \underline{\lim}_\delta \, q_{B^*}(F_\delta) \\
&\leq M
\end{aligned}
$$

$$
\Longrightarrow p_{B^*}\left(\sum_{j=1}^{n} Jx_j\right) \leq M, \quad \forall\, n \geq 1.
$$

or, $$q_{B^*}(F) \leq M$$

and hence (a) follows.

For proving (b), it is sufficient to show that $F_\delta \to F$ in $(\tilde{X}, \tau^*)$ in view of (*) and (a). Therefore consider a $\beta(X_1^*, X)$-bounded subset B^* of X_γ^*. Then by (*) and(a)

$$q_{B^*}(F_\delta - F) \leq 2M, \quad \forall \quad \delta \in \Lambda$$

and so

$$p_{B^*}(Jx_j^\delta - Jx_j) \leq 4M, \quad \forall\, \delta \in \Lambda \ , \ j \geq 1.$$

Now for given $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and a δ_0 in Λ depending on ε and N such that

$$\sum_{j>N} \frac{1}{2^j} < \frac{\varepsilon}{8M}$$

and

$$p_{B^*}(Jx_j^{\delta}-Jx_j) < \frac{\varepsilon}{2N}, \ \forall\ \delta \geq \delta_o \text{ and } 1 \leq j \leq N.$$

Then

$$\hat{q}_{B^*}(F_\delta - F) = \sum_{j\geq 1} \frac{p_{B^*}(Jx_j^{\delta}-Jx_j)}{2^j}$$

$$\leq \sum_{j=1}^{N} \frac{p_{B^*}(Jx_j^{\delta}-Jx_j)}{2^j} + 4M \sum_{j>N} \frac{1}{2^j}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \ \forall\ \delta > \delta_o$$

Consequently, $F_\delta \rightarrow F$ in $(\tilde{X}, \tau^*)$ and so (b) is proved. Hence $\tilde{X}_b$ is γ-complete.

Finally, we have

<u>Proposition</u> 3.7 : Let (X,T) be also sequentially complete. Then the space $\tilde{X}_b \equiv (\tilde{X}, \tau, \tau^*)$ is the canonical γ-completion of $X_b \equiv (X,T,T^*)$.

<u>Proof</u> : For $x \in X$, let us note that the series $\sum_{j\geq 1} Jx_j$, where $x = \sum_{j\geq 1} x_j$ in (X,T), $\sigma(X_{\gamma 1}^*, X_\gamma^*)$ converges by the T-$\beta(X_{\gamma 1}^{**}, X_\gamma^*)|_{JX}$-continuity of J. Thus we can define a map $E : X \rightarrow \tilde{X}$ by

$$E(x) = \{Jx_j\}$$

where $x \in X$ with $x = \sum_{j\geq 1} x_j$, $x_j \in M_j$, $j \geq 1$. Since J is linear, E is clearly linear and one-to-one for

$$E(x) = 0 \implies \{Jx_j\} = 0$$

$$\implies Jx_j = 0, \ \forall\ j \geq 1$$

$$\implies x_j = 0, \ \forall\ j \geq 1$$

$$\implies x = 0.$$

Now in order to prove the result, we need show

(i) $\tilde{X}_b \equiv (\tilde{X}, \tau, \tau^*)$ is normal and γ-complete;

(ii) $E : (X,T) \to (\tilde{X},\tau)$ and $E : (X,T^*) \to (\tilde{X},\tau^*)$

are topological isomorphisms into; and

(iii) $E(X)$ is γ-dense in $\tilde{X}$

We have already proved (i) in Propositions 3.5 and 3.6.

For proving (ii), let us first show the T-τ continuity of E. Therefore, consider a seminorm q_{B^*} of τ and x in X with $x = \sum_{j\geq 1} x_j$, $x_j \in M_j$, $j \geq 1$. Then by T-$\beta(X^{**}_{\gamma 1}, X^*_\gamma)|_{JX}$-continuity of J there exist $M \in \mathbb{R}_+$ and $p \in \mathcal{D}_T$ such that

$$p_{B^*}(Jx) \leq M\, p(x),\ \forall\, x \in X.$$

Consequently,

$$\begin{aligned} q_{B^*}(Ex) &= q_{B^*}(\{Jx_j\}) \\ &= \sup_n p_{B^*}\Big(\sum_{j=1}^n Jx_j\Big) \\ &\leq M \sup_n p\Big(\sum_{j=1}^n x_j\Big) = M\, \bar{p}(x). \end{aligned}$$

Since the topologies T and $\bar{T}$ are equivalent by Proposition 1.3 the T-τ continuity of E follows.

For showing the τ-T continuity of E^{-1}, consider a semino $p \in \mathcal{D}_T$. By Theorem 2.3, there exists a seminorm $p_{B_1^*}$ of $\beta(X^{**}_{\gamma 1},$ such that

(**) $\qquad p(x) \leq p_{B_1^*}(Jx),\ \forall\, x \in X$

Now if $x = \sum_{j\geq 1} x_j$, $x_j \in M_j$, $j \geq 1$, then

$$p_{B_1^*}(Jx) = \lim_{n\to\infty} p_{B_1^*}\left(\sum_{j=1}^{n} Jx_j\right)$$

$$\Longrightarrow p(x) \leq \sup_n p_{B_1^*}\left(\sum_{j=1}^{n} Jx_j\right) = q_{B_1^*}(Ex).$$

Thus E is a T-τ-topological isomorphism from X into $\tilde{X}$.

Now, for $T^*-\tau^*$ continuity of E, note that for each $\hat{q}_{B^*}$,

$$\hat{q}_{B^*}(Ex) = \sum_{j\geq 1} \frac{p_{B^*}(Jx_j)}{2^j}$$

$$\leq M \sum_{j\geq 1} \frac{p(x_j)}{2^j} = M\ p^*(x)$$

for some $p \in \mathcal{D}_T$ and some $M \in \mathbb{R}_+$ by (*).

The τ^*-T^* continuity of E^{-1} follows from (**), for we have

$$p^*(x) = \sum_{j\geq 1} \frac{p(x_j)}{2^j}$$

$$\leq \sum_{j\geq 1} \frac{p_{B_1^*}(Jx_j)}{2^j} = \hat{q}_{B_1^*}(Ex)$$

for x in X with $x = \sum_{j\geq 1} x_j$. Hence E is a $T^*-\tau^*$ topological isomorphism.

To prove (iii), consider an $F = \{Jx_j\} \in \tilde{X}$. Then it suffice to show that there exists a sequence $\{F_n\}$ in E(X) such that $F_n \xrightarrow{\gamma} F$ in $\tilde{X}_b$. For $n \geq 1$, define

$$F_n = \{Jx_1, \ldots, Jx_n, 0, 0, \ldots\}.$$

Then $F_n \in E(X)$, for each $n \geq 1$ since $F_n = E(\sum_{j=1}^{n} x_j)$ and $\sum_{j=1}^{n} x_j \in X$

Also the sequence $\{F_n\}$ is τ-bounded, for

$$q_{B^*}(F_n) = \sup_{1\leq k\leq n} p_{B^*}(\sum_{j=1}^{k} Jx_j)$$

$$\leq q_{B^*}(F), \forall n \geq 1.$$

As $\sum_{j\geq 1} \frac{p_{B^*}(Jx_j)}{2^j}$ is convergent for each $\beta(X_1^*,X)|_{X_\gamma^*}$-bounded subset B^*, for given $\varepsilon > 0$ there exists n_o depending on B^* and ε, such that

$$\sum_{j\geq n+1} \frac{p_{B^*}(Jx_j)}{2^j} < \varepsilon, \forall n \geq n_o.$$

Consequently,

$$\hat{q}_{B^*}(F_n-F) < \varepsilon, \forall n \geq n_o.$$

Thus $E(X)$ is γ-sequentially dense in $\tilde{X}_b$ and the result is completely proved.

4. Relationship Between k-Reflexivity and γ-Reflexivity :

In this section we prove results which lead us to establish the relationship between k- and γ-reflexivities of a canonical bi-l.c. TVS and deduce a known result in the basis theory. As mentioned earlier, (X,T) is a Mazur, infrabarrelled space with e-S.D. $\{M_n;P_n\}$, and the γ-dual $(X_\gamma^*, \beta(X_1^*,X)|_{X_\gamma^*})$ of the corresponding canonical bi-l.c. TVS is a Mazur infrabarrelled space. Let us also recall the map E introduced in the Propositi 3.7, namely $E : X \to \tilde{X}$, with $E(x) = \{Jx_j\}$, $x = \sum_{j\geq 1} x_j, x_j \in M_j, j \geq$

Define $e : \tilde{X} \to X_{\gamma_1}^{**}$ by

$$e(\{Jx_j\}) = \sum_{j\geq 1} Jx_j, \ \forall \{Jx_j\} \in \tilde{X}.$$

the convergence of the series being relative to $\sigma(X^{**}_{\gamma 1}, X^{*}_{\gamma})$.

In terms of the map e, we characterize k-reflexivity in

Proposition 4.1 : For a sequentially complete space (X,T), the canonical bi-l.c. TVS $X_b \equiv (X,T,T^*)$ is k-reflexive if and only if e is onto.

Proof : Let X_b be k-reflexive. Then by Proposition 2.8 each M_j is reflexive. For showing that e is onto, let us consider an F in $X^{**}_{\gamma 1}$. Since $\{P^{**}_j(X^{**}_{\gamma 1}); P^{**}_j\}$ is a $\sigma(X^{**}_{\gamma 1}, X^{*}_{\gamma})$-S.D. of $X^{**}_{\gamma 1}$, we have

$$F = \sum_{j\geq 1} P^{**}_j(F) \text{ in } \sigma(X^{**}_{\gamma 1}, X^{*}_{\gamma}).$$

Since $J(M_j) = M^{**}_j$ by reflexivity of M_j and the map ϕ_j in Proposition 2.7 is identity on JM_j, it follows that $J(M_j) = P^{**}_j(X^{*}_{\gamma}$
Hence there exist $x_j \in M_j$, $j \geq 1$ such that

$$P^{**}_j(F) = Jx_j, \ j \geq 1$$

Therefore

$$F = \sum_{j\geq 1} Jx_j \text{ in } \sigma(X^{**}_{\gamma 1}, X^{*}_{\gamma}).$$

$$\Longrightarrow F = e(\{Jx_j\}), \text{ that is, e is onto.}$$

Conversely, let e be onto. Then for $F \in X^{**}_{\gamma\gamma}$,

$$F = \sum_{j\geq 1} Jx_j \text{ in } \sigma(X^{**}_{\gamma 1}, X^{*}_{\gamma}).$$

Also, by Corollary 5.3.7

$$F = \sum_{j\geq 1} P_j^{**}(F) \text{ in } \beta(X_{\gamma 1}^{**}, X_\gamma^*)$$

and so in $\sigma(X_{\gamma 1}^{**}, X_\gamma^*)$. Hence

$$(*) \qquad F = \sum_{j\geq 1} Jx_j = \sum_{j\geq 1} P_j^{**}(F).$$

Now fix j_o in $\mathbb{N}$. Then for f in X_γ^*, we have from (*)

$$(\sum_{j\geq 1} Jx_j)(P_{j_o}^* f) = (\sum_{j\geq 1} P_j^{**}(F))(P_{j_o}^* f)$$

$$\Longrightarrow \sum_{j\geq 1} (P_{j_o}^* f)(x_j) = \sum_{j\geq 1} F(P_j P_{j_o}^*(f))$$

$$\Longrightarrow f(x_{j_o}) = F(P_{j_o}^* f)$$

$$\Longrightarrow (Jx_{j_o})(f) = (P_{j_o}^{**} F)(f)$$

Since $f \in X_\gamma^*$ and j_o in $\mathbb{N}$ are arbitrary, we get

$$Jx_j = P_j^{**}(F) \;\forall\; j \geq 1.$$

$\Longrightarrow JM_j = P_j^{**}(X_{\gamma\gamma}^{**})$, $\forall\, j \geq 1$, that is, J is an onto map from M_j to $P_j^{**}(X_{\gamma\gamma}^{**})$. Therefore, applying Proposition 2.7 again, J maps M_j onto M_j^{**}. Now invoking the proof of Proposition 2.8, J maps X onto $X_{\gamma\gamma}^{**}$, that is, X_b is k-reflexive.

As an immediate consequence of the preceding result, we have

<u>Proposition</u> 4.2 : If the S.D. of a sequentially complete space (X,T) is shrinking, then $e : \tilde{X} \to X_{11}^{**}$ is onto if and only if each M_j is reflexive.

<u>Proof</u> : Since the S.D. is shrinking, by Corollary 5.3.4, $X_\gamma^* = X_1^*$ and therefore $X_{\gamma 1}^{**} = X_{11}^{**}$. Now the result follows immediately from

the preceding proposition and Proposition 2.8.

For our next result concerning the characterization of boundedly complete S.D., we need

Lemma 4.3. If (X,T) is sequentially complete, then

$$E(X) = \{\{Jx_j\} : x_j \in M_j,\ j \geq 1 \text{ and } \sum_{j\geq 1} Jx_j \text{ converges in } \beta(X_{\gamma 1}^{**}, X_\gamma^*)\}.$$

Proof : In view of T-$\beta(X_{\gamma 1}^{**}, X_\gamma^*)|_{JX}$-continuity of J, it is sufficient to show the existence of a point x in X for each sequence $\{x_j\}$ with $x_j \in M_j$, $j \geq 1$ and $\sum_{j\geq 1} Jx_j$ converges in $\beta(X_{\gamma 1}^{**}, X_\gamma^*)$ such that $x_j \to x$ in (X,T). Therefore, consider such a sequence $\{Jx_j\}$. Then $\{\sum_{j=1}^{n} Jx_j : n \geq 1\}$ is a $\beta(X_{\gamma 1}^{**}, X_\gamma^*)$-Cauchy sequence in $X_{\gamma 1}^{**}$. Applying the inequality (2.4) in the proof of the Theorem 2.3, it follows that $\{\sum_{j=1}^{n} x_j : n \geq 1\}$ is a T-Cauchy sequence in (X,T). Hence there exists an x in X such that $x = \sum_{j=1} x_j$. This completes the proof.

Theorem 4.4 : The S.D. $\{M_n; P_n\}$ for a sequentially complete space (X,T) is boundedly complete if and only if E is onto.

Proof : Let $\{M_n\}$ be boundedly complete. Then $\{JM_n\}$ is also boundedly complete for $(JX, \beta(X_{\gamma 1}^{**}, X_\gamma^*)|_{JX})$; indeed, if $\{\sum_{j=1}^{n} Jx_j : n \geq 1\}$ is $\beta(X_{\gamma 1}^{**}, X_\gamma^*)$-bounded, then by the inequality (2.4), $\{\sum_{j=1}^{n} x_j : n \geq 1\}$ is T-bounded and therefore there exists

an x in X such that $x = \sum_{j\geq 1} x_j$ in (X,T). Consequently, by $T-\beta(X_{\gamma 1}^{**}, X_\gamma^*)|_{JX}$ continuity of J, $Jx = \sum_{j\geq 1} Jx_j$ in $\beta(X_{\gamma 1}^{**}, X_\gamma^*)$.

To prove the equality $\tilde{X} = E(X)$, we need show that $\tilde{X} \subset E(X)$ as the other inclusion is trivially true. Therefore, consider $\{Jx_j\} \in \tilde{X}$. Consequently, $\{\sum_{j=1}^{n} Jx_j : n \geq 1\}$ is $\beta(X_{\gamma 1}^{**}, X_\gamma^*)$-bounded by the note following Proposition 3.2. Thus $\sum_{j\geq 1} Jx_j$ converges in $\beta(X_{\gamma 1}^{**}, X_\gamma^*)$ by the above arguments and hence $\{Jx_j\} \in E(X)$.

For converse, let us consider a T-bounded sequence $\{\sum_{j=1}^{n} x_j : n \geq 1\}$, $x_j \in M_j$, $j \geq 1$. Therefore, $\{\sum_{j=1}^{n} Jx_j : n \geq 1\}$ is $\beta(X_{\gamma 1}^{**}, X_\gamma^*)$-bounded by Theorem 2.3. Consequently, $\{Jx_j\} \in \tilde{X}=E(X)$. Hence there exists an x in X such that $x = \sum_{j\geq 1} x_j$. Thus $\{M_n\}$ is boundedly complete and the result is completely established.

An immediate consequence of this result is

<u>Corollary</u> 4.5 : For a sequentially complete l.c. TVS (X,T), X_b is γ-complete if and only if $\tilde{X} = E(X)$.

<u>Proof</u> : Follows immediately from Propositions 5.3.16 and 4.4.

We are now prepared to prove the main result of this section, namely

<u>Theorem</u> 4.6 : If (X,T) is sequentially complete, then X_b is γ-reflexive if and only if X_b is k-reflexive and γ-complete.

<u>Proof</u> : Let X_b be γ-reflexive. Then $J : X \to X_{\gamma 1}^{**}$ is onto. Consequently, $J : X \to X_{\gamma\gamma}^{**}$ is onto. Hence X_b is k-reflexive.

For showing the γ-completeness of X_b, it is enough to show that $\tilde{X} \subset E(X)$ in view of Corollary 4.5. So, consider $\{Jx_j\} \in \tilde{X}$. Then $\sum_{j\geq 1} Jx_j$ converges in $\sigma(X_{\gamma 1}^{**}, X_{\gamma}^{*})$ to some element, say F, in $X_{\gamma 1}^{**}$. Hence by the γ-reflexivity of X_b, there exists a y in X such that

$$(*) \qquad F = Jy$$

Since $y = \sum_{j\geq 1} y_j$, $y_j \in M_j$, $j \geq 1$ relative to the topology T, applying Proposition 2.3, we have

$$Jy = \sum_{j\geq 1} Jy_j$$

the convergence of the series being relative to $\beta(X_{\gamma 1}^{**}, X_{\gamma}^{*})$ and hence also relative to $\sigma(X_{\gamma 1}^{**}, X_{\gamma}^{*})$. Thus

$$\sum_{j\geq 1} Jx_j = \sum_{j\geq 1} Jy_j.$$

Now following the arguments as in the proof of the converse part of Proposition 4.1, we obtain

$$x_j = y_j, \ \forall\, j \geq 1$$

Consequently, $\sum_{j\geq 1} Jx_j$ converges in $\beta(X_{\gamma 1}^{**}, X_{\gamma}^{*})$. Hence by Lemma 4.3, $\{Jx_j\} \in E(X)$ and $\tilde{X} \subset E(X)$.

Conversely, let X_b be k-reflexive and γ-complete. For showing γ-reflexivity of X_b, consider an F in $X_{\gamma 1}^{**}$. Then applying Proposition 4.1,

$$F = e(\{Jx_j\}) = \sum_{j\geq 1} Jx_j$$

For some $\{Jx_j\} \in \tilde{X}$. Also by Theorem 4.4 $\tilde{X} = E(X)$ and so there exists an $x \in X$ such that,

$$Jx = \sum_{j\geq 1} Jx_j.$$

Thus $F = Jx$ for some $x \in X$ and hence X_b is γ-reflexive.

As an immediate consequence, we have

Corollary 4.6 : Let (X,T) be a sequentially complete infrabarrelled space with a Schauder basis $\{x_i;f_i\}$ such that $(X_\gamma^*,\ \beta(X_1^*,X)|_{X_\gamma^*})$ is a barrelled space. Then X_b is γ-reflexive if and only if X_b is γ-complete.

Proof : By Propositions 1.2.20 and 1.3.5, (X,T) is a Mazur space. Also each M_j being one dimensional space, is reflexive for $j \geq 1$. Hence the result follows immediately from Theorem 4.6.

Finally, we deduce the following result due to Retherford ([103], Theorem 2.3, p. 281) by using the techniques of bi-locally convex spaces as follows.

Theorem 4.7 : If (X,T) is a barrelled, semireflexive (and hence reflexive) complete space with a Schauder basis $\{x_i;f_i\}$, then $\{x_i;f_i\}$ is both shrinking and boundedly complete.

Proof : Since (X,T) is reflexive, it follows from Proposition 4.3.6 that X_b is γ-reflexive and saturated. Hence, $X_\gamma^* = X_1^*$ and s $(X_\gamma^*,\ \beta(X_1^*,X)|_{X_\gamma^*})$ is barrelled by Proposition 1.2.27. Consequently, X_b is γ-complete by Corollary 4.6. Now the applications of Corollaries 5.3.4 and 5.3.16 yield the shrinking and boundedly complete character of $\{x_i;f_i\}$.

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